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First order Galilean superfluid dynamics

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We study dynamics of an (anomalous) Galilean superfluid up to first order in derivative expansion, both in parity-even and parity-odd sectors. We construct a relativistic system—null superfluid, which is a null fluid (introduced in N. Banerjee, S. Dutta, and A. Jain Akash, [Phys. Rev. D **93**, 105020 (2016).]) with a spontaneously broken global U(1) symmetry. A null superfluid is in one-to-one correspondence with a Galilean superfluid in one lower dimension; i.e., they have the same symmetries, thermodynamics, constitutive relations and are related to each other by a mere choice of basis. The correspondence is based on null reduction, which is known to reduce the Poincaré symmetry of a theory to Galilean symmetry in one lower dimension. To perform this analysis, we use off-shell formalism of (super)fluid dynamics, adopting it appropriately to null (super)fluids. We also verify these results via $c \to \infty$ limit of a parent relativistic system.

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I. INTRODUCTION AND SUMMARY

Hydrodynamics is an effective description of low energy fluctuations of a quantum system around thermodynamic equilibrium. In this description, we assume the hydrodynamic system, known as a fluid, to be at a finite temperature, and study its fluctuations at length scales much larger than the mean free path of the system. In this limit and far away from any second order phase transition point, a fluid can be described by a small number of degrees of freedom known as hydrodynamic modes: temperature, chemical potential(s) and normalized fluid velocity. Various conserved currents of the system can then be written in terms of these hydrodynamic modes, arranged as a perturbative expansion in derivatives, known as fluid constitutive relations. At any particular order in derivative expansion, constitutive relations contain all the possible independent tensor structures allowed by symmetry at that order, multiplied with unknown coefficients known as transport coefficients. If the underlying quantum theory has a continuous global symmetry which is spontaneously broken in the ground state, then the low energy fluctuations can contain massless Goldstone modes corresponding to the broken symmetry. Therefore for fluids with a spontaneously broken symmetry, known as superfluids, hydrodynamic modes also contain these Goldstone modes. This leads to a considerable modification of the constitutive relations, adding new tensor structures containing the derivatives of the Goldstone modes and hence new transport coefficients. In this paper, we work out the most generic constitutive relations of a Galilean superfluid up to first order in the derivative expansion.

Superfluidity was first observed in liquid helium by [1,2] in 1938, while studying its flow through a thin capillary. They observed that liquid helium flows through the capillary without any dissipation, hence inspiring the name "superfluid." Other than this dissipationless flow, superfluids have many more striking features, such as upon rotation they develop vortices (quasi-one-dimensional strings whose number is proportional to the externally imposed angular momentum). Furthermore, their specific heat shows a sudden change in behavior at a certain critical temperature. Above the critical temperature system behaves like an ordinary fluid, though as the temperature drops below the critical temperature, system undergoes a phase transition from the ordinary fluid phase to the superfluid phase.

Study of superfluid dynamics has been a topic of interest for a long time. First theory of superfluid dynamics was written down by London [3] in 1938, followed by a two-fluid model of superfluids proposed by Landau and Tisza [4,5] in 1940s. They studied ideal superfluids in a non-relativistic setting, which was later generalized to describe a relativistic superfluid by [6–11]. The subject was recently revisited by [12–14] (see also [15]), who rederived the relativistic superfluid constitutive relations using the second law of thermodynamics and equilibrium partition function. Among other interesting results, they found that up to first order in derivative expansion, a relativistic

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TABLE I. Counting of the independent first-order transport coefficients consistent with the second law of thermodynamics. The numbers with a "tilde" represent the parity-odd count (in three spatial dimensions) while the "un-tilde" numbers are the parity-even count. The coefficients with an "asterisk" drop out on imposing Onsager relations (microscopic time-reversal invariance). Finally, in the last row we have given the number of undetermined constants including the anomaly constant. In both relativistic and Galilean cases, we have gotten rid of a hydrostatic coefficient by redefinition of the U(1) phase φ.

	Relativistic fluid	Relativistic superfluid	Galilean fluid	Galilean superfluid
Hydrostatic	$0+\tilde{0}$	$2+\tilde{2}$	$0+\tilde{0}$	$3+\tilde{3}$
Nonhydrostatic nondissipative	$0+\tilde{0}$	$7^{*}+\tilde{4}$	$1^{*}+\tilde{0}$	$13^{*}+\tilde{7}$
Dissipative	$3+\tilde{0}$	$14+\tilde{1}^{*}$	$5+\tilde{0}$	$22+\tilde{3}^{*}$
Total	$3+\tilde{0}=3$	$23+\tilde{7}=30$	$6+\tilde{0}=6$	$38+\tilde{13}=51$
Total (with Onsager)	$3+\tilde{0}=3$	$16+\tilde{6}=22$	$5+\tilde{0}=5$	$25+\tilde{10}=35$
Hydrostatic constants	$\tilde{3}+\tilde{1}_{\text{anomaly}}$	$\tilde{1}+\tilde{1}_{\text{anomaly}}$	$\tilde{4}+\tilde{1}_{\text{anomaly}}$	$\tilde{1}+\tilde{1}_{\text{anomaly}}$

superfluid is characterized by pressure (at ideal order), 23 parity-even and 7 parity-odd first-order transport coefficients and two undetermined constants including the anomaly constant (after imposing Onsager relations and CPT invariance these numbers drop down to 16 parity-even and 6 parity-odd transport coefficients and one anomaly constant). See Table I for a summary and Sec. II for more details.

In this paper, we perform a similar exercise for Galilean superfluids. We derive the constitutive relations for a Galilean superfluid consistent with the second law of thermodynamics, up to first order in derivative expansion, both in parity even and odd sectors. Study of Galilean superfluids is important because it provides a laboratory to probe many-body physics in extreme quantum regime with high-precision [16]. Relativistic effects are important in high-energy superfluids, where mass of the constituents is small compared to their kinetic energy, e.g. quark superfluidity in compact stars. In contrast, for low-energy systems such as liquid helium and ultracold atomic gases, a Galilean framework is more ideal.

Recently in [17,18], we established that one can derive the most generic constitutive relations for an ordinary Galilean fluid starting from a relativistic system, namely a null fluid in one higher dimension, followed by a null reduction.[1] [20,21]. Loosely speaking, null fluid is a fluid coupled to a background with fields: a metric g_{MN}, a U(1) gauge field A_M and a covariantly constant null isometry $\mathcal{V}=\{V^M,\Lambda_V\}$ with $V^M A_M+\Lambda_V=\text{constant}$. We call this background a null background.[2] Theories on a null background, which we call null theories, are demanded to be invariant under $\mathcal{V}$ preserving diffeomorphisms and gauge transformations. Upon performing null reduction, i.e. choosing a basis $\{x^M\}=\{x^-,t,x^i\}$ such that $\mathcal{V}=\{V=\partial_-,\Lambda_V=0\}$, these restricted transformations reduce to the well known Galilean transformations on the background spanned by coordinates $\{t,x^i\}$. It suggests that null theories are entirely equivalent to Galilean theories, and are related by merely this choice of basis. It follows that a fluid on null background—null fluid—is entirely equivalent to a Galilean fluid. Their constitutive relations, conservation laws, thermodynamics etc. match exactly to all orders in derivative expansion. Due to presence of an additional vector field V^M, constitutive relations of a null fluid are vastly different from those of a relativistic fluid and contain many more transport coefficients. This accounts for the additional transport coefficients in a Galilean fluid as compared to a relativistic fluid,[3] while at the same time establishing that the most generic Galilean fluid cannot be gained by null reduction of an ordinary relativistic fluid.

In this paper we take the construction of null fluids one step further to include null superfluids, i.e. we construct a null fluid with a spontaneously broken U(1) symmetry. The corresponding Goldstone mode is a new field in the theory and modifies the constitutive relations of an ordinary null fluid. Once we have the constitutive relations for a null superfluid, corresponding Galilean superfluid constitutive relations follow trivially via null reduction. We find that up to first order in derivatives, a Galilean superfluid is described by pressure P (at ideal order), a total of 51 first-order transport coefficients and two unknown constants including the anomaly constant. Out of these 51 coefficients, 38 lie in parity-even sector while 13 are in

[1]Null reduction of an ordinary relativistic fluid gives us a constrained Galilean fluid as found in [19].

[2]Here, definition of null backgrounds has been adapted to a torsionless spacetime. For backgrounds with torsion, look at [22].

[3]The reader might wonder how a Gailean (super)fluid can have more transport coefficients than a relativsitic one. Though the Galilean symmetry has more generators than Poincaré symmetry (accounting for the additional mass conservation operator), a Galilean system has an additonal U(1) mass current in its spectrum. Therefore the most generic Galilean (super)fluid can admit more transport coefficients than a relativistic (super)fluid. More discussion can be found in Sec. V.

parity-odd sector. Furthermore, only 22 parity-even and 3 parity-odd coefficients are dissipative. Out of the nondissipative coefficients, 3 parity-even and 3 parity-odd coefficients describe equilibrium physics, while the remaining 13 parity-even and 7 parity-odd coefficients describe nondissipative effects away from equilibrium. Finally, following the intuition from relativistic superfluids and known Galilean results in [23], there are hints that the 7 parity-even nondissipative, nonhydrostatic coefficients and 3 parity-odd dissipative coefficients are switched off using Onsager relations (imposing microscopic reversibility of field theories). This would imply that the parity-odd sector is purely nondissipative. However, a detailed microscopic calculation is required to establish confidence in these Galilean Onsager relations, which we do not perform in this paper. In Table I, we have summarized the counting of transport coefficients for the most generic Galilean superfluid, along with a comparison with relativistic superfluids reviewed in Sec. II and known results for ordinary Galilean and relativistic fluids.

Another recent development in hydrodynamics is offshell formalism introduced by [24–26], which streamlines the analysis of constitutive relations in accordance with the second law of thermodynamics, up to arbitrarily high orders in derivative expansion. We have reviewed this formalism in Sec. II. In a nutshell, for ordinary fluids the formalism requires us to consider a version of the second law of thermodynamics which is valid for thermodynamically nonisolated fluids,

$$\nabla_\mu J_S^\mu + \frac{u_\mu}{T}(\nabla_\nu T^{\mu\nu} - F^{\mu\rho}J_\rho - \mathrm{T}_{\mathrm{H}}^{\mu\perp}) + \frac{\mu}{T}(\nabla_\mu J^\mu - \mathrm{J}_{\mathrm{H}}^{\perp}) = \Delta \geq 0. \tag{1.1}$$

Since the fluid is not thermodynamically isolated, it is allowed to interact with its surrounding and hence conservation laws are no longer satisfied. Therefore the original second law $\nabla_\mu J_S^\mu \geq 0$ has been modified with combinations of the conservation laws. We need to find the most generic constitutive relations for $T^{\mu\nu}$, J^μ allowed by symmetries (modulo terms related to each other by conservation laws) which satisfy Eq. (1.1) for some entropy current J_S^μ and $\Delta \geq 0$. When we move to superfluids, we have an additional field φ (the Goldstone mode) which comes with its own equation of motion $K = 0$, the Josephson equation. Going offshell in φ, conservation equations get modified by combinations of K, and the second law of thermodynamics for thermodynamically nonisolated superfluids takes the form (see [27] for more details),

$$\nabla_\mu J_S^\mu + \frac{u_\mu}{T}(\nabla_\nu T^{\mu\nu} - F^{\mu\rho}J_\rho - \mathrm{T}_{\mathrm{H}}^{\mu\perp} - \xi^\mu K) + \frac{\mu}{T}(\nabla_\mu J^\mu - \mathrm{J}_{\mathrm{H}}^{\perp} + K) = \Delta \geq 0. \tag{1.2}$$

Note that contrary to the philosophy of [24–26], though we have gone offshell in φ we have not modified the second law with a multiple of K. Rather, we require the second law of thermodynamics to be satisfied even for offshell configurations of φ. Next, we find the most generic "superfluid constitutive relations" $T^{\mu\nu}, J^\mu$, K allowed by symmetries (modulo terms related to each other by conservation laws or the Josephson equation) which satisfy Eq. (1.2) for some entropy current J_S^μ and $\Delta \geq 0$. In Sec. III, we have extended this formalism to null (super)fluids, and used it to work out the constitutive relations of a null/Galilean superfluid up to first order in derivative expansion.

The paper is organized as follows: we start Sec. II with a review of offshell formalism for relativistic hydrodynamics. Readers well familiar with this formalism can skip to Sec. II B, where we have reviewed offshell formalism for relativistic superfluids and later used it to work out respective constitutive relations up to first order in derivative expansion. Next in Sec. III, we introduce offshell formalism for null superfluids and find respective constitutive relations up to first order in derivative expansion. The null superfluid results have been reduced to Galilean superfluids in Sec. IV. In Sec. V, we have argued how these results can also be obtained by $c \to \infty$ limit of a parent relativistic theory. These are the main results of this paper. Finally, we conclude with some discussion in Sec. VI. The paper contains three Appendices: in Appendix A we give a detailed derivation of first-order constitutive relations of a relativistic superfluid in offshell formalism, and in Appendix B we present equilibrium partition function for null superfluids. Finally, in Appendix C, we give details of some computations glossed over in the main text.

II. REVISITING RELATIVISTIC SUPERFLUIDS

Before going to null superfluids, it is instructive to revisit the relativistic superfluids first. It will help us appreciate the similarities between the two systems, while at the same time allowing for an isolation of the differences. Needless to say, all the results in this section have already been worked out in the literature [12–14]; however, our approach will be slightly different. We will work in the "off-shell formalism of hydrodynamics," which was introduced for ordinary (nonsuper) fluids in [24,26], and later extended to superfluids in [27].

A. Off-shell formalism for relativistic ordinary fluids

Let us begin with ordinary relativistic fluids. Consider a d-dimensional manifold $\mathcal{M}_d$ equipped with the background fields: a metric $g_{\mu\nu}$ and a U(1) gauge field A_μ. Physical theories coupled to $\mathcal{M}_d$ are required to be invariant under diffeomorphisms and U(1) gauge transformations. These act on the said background fields as

$$\delta_\chi g_{\mu\nu} = \pounds_\chi g_{\mu\nu} = \nabla_\mu \chi_\nu + \nabla_\nu \chi_\mu,$$
$$\delta_\chi A_\mu = \pounds_\chi A_\mu + \partial_\mu \Lambda_\chi = \partial_\mu(\Lambda_\chi + \chi^\nu A_\nu) + \chi^\nu F_{\nu\mu}, \tag{2.1}$$

for some diffeomorphism and U(1) gauge parameters $\mathcal{X} = \{\chi^\mu, \Lambda_\chi\}$ respectively. In this work we will only be interested in a particular class of these theories—*fluids*, which are the universal near equilibrium limit of quantum field theories. Near equilibrium, the spectrum of any quantum field theory on $\mathcal{M}_d$ must contain an energy momentum tensor $T^{\mu\nu}$ and a charge current J^μ. These quantities satisfy a set of conservation laws (here ∇_μ is the covariant derivative associated with $g_{\mu\nu}$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength associated with A_μ and $\mathrm{T}^{\mu\perp}_{\mathrm{H}}, \mathrm{J}^{\perp}_{\mathrm{H}}$ are Hall currents carrying the anomalous contribution to the conservation equations),

$$\nabla_\mu T^{\mu\nu} - F^{\nu\rho} J_\rho - \mathrm{T}^{\nu\perp}_{\mathrm{H}} = 0, \qquad \nabla_\mu J^\mu - \mathrm{J}^{\perp}_{\mathrm{H}} = 0, \tag{2.2}$$

provided that the system is thermodynamically isolated. In fact, Eq. (2.2) can be taken as a definition of thermodynamic isolation for near equilibrium quantum systems. The conservation laws Eq. (2.2) can also be thought of as a "near equilibrium version" of the first law of thermodynamics, which imposes the conservation of not just energy, but also momentum and U(1) charge. Formally, we define an (ordinary) fluid as a near equilibrium system characterized by the currents $T^{\mu\nu}$, J^μ, with dynamics given by the conservation laws Eq. (2.2) imposed as the "equations of motion." Since Eq. (2.2) are $(d+1)$ equations in d dimensions, they can provide dynamics for a fluid described by an arbitrary set of $(d+1)$ variables. We choose these to be a normalized fluid velocity u^μ (with $u^\mu u_\mu = -1$), a temperature T and a chemical potential μ, collectively known as the hydrodynamic fields (modes). A fluid hence is completely characterized by a gauge-invariant expression of $T^{\mu\nu}$, J^μ in terms of $g_{\mu\nu}$, A_μ, u^μ, T, μ, known as the hydrodynamic constitutive relations. The near equilibrium assumption allows us to arrange these constitutive relations as a perturbative expansion in derivatives (known as derivative or gradient expansion), consistently truncated at a finite order in derivatives.

Being a thermodynamic system, a fluid is also required to satisfy a version of the second law of thermodynamics. It states that there must exist an entropy current J^μ_S whose divergence is positive semidefinite everywhere, i.e.,

$$\nabla_\mu J^\mu_S = \Delta \geq 0, \tag{2.3}$$

as long as the fluid is thermodynamically isolated (i.e. conservation laws Eq. (2.2) or equivalently the first law(s) of thermodynamics are satisfied). The job of hydrodynamics now is to find the most general constitutive relations $T^{\mu\nu}$, J^μ and an associated J^μ_S, Δ order by order in derivative expansion, such that Eq. (2.3) is satisfied for thermodynamically isolated fluids. This task has been extensively undertaken in the literature [28–33].

The problem stated in this language, however, turns out to be increasingly hard to solve as we go to second or higher orders in derivative expansion [34]. Fortunately, it was realized in [24] that most of the complication in the aforementioned computation comes from the fact that we need to maintain the thermodynamic isolation (i.e. satisfy the conservation equations) perturbatively at every order. A much easier problem to solve is to allow for the fluid to interact with its surroundings, i.e. break the conservation laws Eq. (2.2) by introducing an arbitrary external momentum P^μ_{ext} and a charge Q_{ext} source,

$$\begin{aligned} \nabla_\mu T^{\mu\nu} - F^{\nu\rho} J_\rho - \mathrm{T}^{\nu\perp}_{\mathrm{H}} &= P^\nu_{\mathrm{ext}}, \\ \nabla_\mu J^\mu - \mathrm{J}^{\perp}_{\mathrm{H}} &= Q_{\mathrm{ext}}. \end{aligned} \tag{2.4}$$

The lhs of the second law in Eq. (2.3) will also need to be augmented with an arbitrary combination of P^μ_{ext}, Q_{ext} for the inequality to be satisfied,

$$\begin{aligned} &\nabla_\mu J^\mu_S + \beta_\nu P^\nu_{\mathrm{ext}} + (\Lambda_\beta + A_\mu \beta^\mu) Q_{\mathrm{ext}} = \Delta \geq 0, \\ &\Rightarrow \nabla_\mu J^\mu_S + \beta_\nu (\nabla_\mu T^{\mu\nu} - F^{\nu\rho} J_\rho - \mathrm{T}^{\mu\perp}_{\mathrm{H}}) \\ &\qquad + (\Lambda_\beta + A_\mu \beta^\mu)(\nabla_\mu J^\mu - \mathrm{J}^{\perp}_{\mathrm{H}}) = \Delta \geq 0, \end{aligned} \tag{2.5}$$

for some fields $\mathcal{B} = \{\beta^\mu, \Lambda_\beta\}$. This version of the second law is known as the off-shell second law of thermodynamics, because the conservation laws, which are imposed as equations of motion on the fluid, are not required to be satisfied. Equation (2.5) can be recast into a yet another useful form by defining a free-energy current G^μ as

$$\begin{aligned} -\frac{G^\mu}{T} &= N^\mu = J^\mu_S + \beta_\nu T^{\mu\nu} + (\Lambda_\beta + A_\nu \beta^\nu) J^\mu, \\ -\frac{\mathrm{G}^{\perp}_{\mathrm{H}}}{T} &= \mathrm{N}^{\perp}_{\mathrm{H}} = \beta_\mu \mathrm{T}^{\mu\perp}_{\mathrm{H}} + (\Lambda_\beta + A_\nu \beta^\nu) \mathrm{J}^{\perp}_{\mathrm{H}}. \end{aligned} \tag{2.6}$$

Having done that, Eq. (2.5) implies a free-energy conservation,

$$\nabla_\mu N^\mu - \mathrm{N}^{\perp}_{\mathrm{H}} = \frac{1}{2} T^{\mu\nu} \delta_{\mathcal{B}} g_{\mu\nu} + J^\mu \delta_{\mathcal{B}} A_\mu + \Delta, \qquad \Delta \geq 0, \tag{2.7}$$

where, similar to Eq. (2.1), we have defined

$$\begin{aligned} \delta_{\mathcal{B}} g_{\mu\nu} &= \pounds_\beta g_{\mu\nu} = \nabla_\mu \beta_\nu + \nabla_\nu \beta_\mu, \\ \delta_{\mathcal{B}} A_\mu &= \pounds_\beta A_\mu + \partial_\mu \Lambda_\beta = \partial_\mu (\Lambda_\beta + \beta^\nu A_\nu) + \beta^\nu F_{\nu\mu}. \end{aligned} \tag{2.8}$$

Recall that the hydrodynamic fields u^μ, T, μ introduced earlier were some arbitrary $(d+1)$ fields chosen to describe the fluid. Like in any field theory, they are permitted to admit an arbitrary redefinition among themselves without changing the physics. This huge amount of freedom can be fixed by explicitly choosing,

$$T = \frac{1}{\sqrt{-\beta^\nu \beta_\nu}}, \qquad u^\mu = \frac{\beta^\mu}{\sqrt{-\beta^\nu \beta_\nu}}, \qquad \mu = \frac{\Lambda_\beta + A_\mu \beta^\mu}{\sqrt{-\beta^\nu \beta_\nu}}, \tag{2.9}$$

or conversely,

$$\beta^\mu = \frac{1}{T} u^\mu, \qquad \Lambda_\beta = \frac{1}{T}\mu - A_\mu \beta^\mu. \tag{2.10}$$

As a consequence of this choice, $\mathcal{B} = \{\beta^\mu, \Lambda_\beta\}$ is just a renaming of the hydrodynamic fields. Finally, we can find the most general gauge-invariant expression of the currents $T^{\mu\nu}$, J^μ in terms of $g_{\mu\nu}$, A_μ, β^μ, Λ_β arranged in a derivative expansion, along with an associated N^μ, Δ such that Eq. (2.7) is satisfied. However, there is a caveat in this way of thinking: these $T^{\mu\nu}$, J^μ are not merely the constitutive relations of a fluid; they also contain information about the external sources $P^\mu_{\rm ext}$, $Q_{\rm ext}$. One way to circumvent this problem is to pick a set of terms which might potentially appear in $T^{\mu\nu}$, J^μ and can be eliminated using the conservation laws, and only consider the solutions $T^{\mu\nu}$, J^μ of Eq. (2.7) (for some N^μ, Δ) which do not involve these terms or their derivatives. $T^{\mu\nu}$, J^μ thus obtained are guaranteed to be the constitutive relations of a fluid, as they will be free from any $P^\mu_{\rm ext}$, $Q_{\rm ext}$ dependence.

Authors in [25,26] illustrated a consistent mechanism to find the most generic constitutive relations of a fluid up to arbitrarily high orders in derivative expansion, which satisfy Eq. (2.7). They further classified these constitutive relations in eight exhaustive classes, which we will not have scope to review here. Instead, in the following subsection, we will review the off-shell analysis of relativistic superfluids which has been introduced in [27], and later adapt it to Galilean superfluids.

B. Off-shell formalism for relativistic superfluids

Let us now review some essential aspects of the off-shell formalism for a relativistic superfluid following the work of [27], and use it to re-derive the respective constitutive relations up to first order in derivative expansion [12–14]. For the sake of brevity, we have pushed the computational details in Appendix A. As we have already mentioned in the introduction, a superfluid is a phase of a fluid where the global U(1) symmetry of the microscopic theory gets spontaneously broken in the ground state due to condensation of a charged scalar operator. The U(1) phase φ of the scalar operator becomes a new field in the theory, along with u^μ, T, μ on which the respective constitutive relations can depend. Under an infinitesimal gauge transformation and diffeomorphism, φ transforms as $\delta_\mathcal{X}\varphi = \chi^\mu \partial_\mu \varphi - \Lambda_\chi$, with covariant derivative,

$$\xi_\mu = \partial_\mu \varphi + A_\mu, \tag{2.11}$$

commonly known as the "superfluid velocity". Just like the dynamics of u^μ, T, μ is given by the conservation equations Eq. (2.2), φ comes with its own equation of motion,[4]

$$K = 0. \tag{2.12}$$

We will be particularly interested in the "off-shell" configurations of the field φ, which we define as the superfluid configurations for which $K \neq 0$. As was suggested by [27], conservation laws for these configurations modify to,

$$\begin{aligned} \nabla_\mu T^{\mu\nu} &= F^{\nu\rho} J_\rho + \mathrm{T}^{\nu\perp}_{\mathrm{H}} + \xi^\nu K, \\ \nabla_\mu J^\mu &= \mathrm{J}^{\perp}_{\mathrm{H}} - K, \end{aligned} \tag{2.13}$$

which trivially turn back to their original form in Eq. (2.2) when $K = 0$. The claim is that "even the φ-offshell configurations of a superfluid satisfy the second law of thermodynamics"; i.e., there exists an entropy current J^μ_S whose divergence is positive semidefinite, i.e.,

$$\nabla_\mu J^\mu_S = \Delta \geq 0, \tag{2.14}$$

as long as the superfluid is thermodynamically isolated (i.e. Eq. (2.13) are satisfied), irrespective of K being zero. Rest of the analysis follows exactly like ordinary fluids; on allowing the superfluid to interact with its surroundings, the second law modifies to,

$$\begin{aligned} &\nabla_\mu J^\mu_S + \beta_\nu(\nabla_\mu T^{\mu\nu} - F^{\nu\rho} J_\rho - \mathrm{T}^{\mu\perp}_{\mathrm{H}} - \xi^\nu K) \\ &\quad + (\Lambda_\beta + A_\sigma \beta^\sigma)(\nabla_\mu J^\mu - \mathrm{J}^\perp_{\mathrm{H}} + K) = \Delta \geq 0. \end{aligned} \tag{2.15}$$

In terms of free-energy current, however, we get,

$$\begin{aligned} \nabla_\mu N^\mu - \mathrm{N}^\perp_{\mathrm{H}} &= \frac{1}{2} T^{\mu\nu} \delta_\mathcal{B} g_{\mu\nu} + J^\mu \delta_\mathcal{B} A_\mu + K \delta_\mathcal{B}\varphi + \Delta, \\ \Delta &\geq 0, \end{aligned} \tag{2.16}$$

where

$$\delta_\mathcal{B}\varphi = \beta^\mu \partial_\mu \varphi - \Lambda_\beta = \frac{1}{T}(u^\mu \xi_\mu - \mu). \tag{2.17}$$

Similar to the ordinary fluid, we should now consider the most generic expressions for $T^{\mu\nu}$, J^μ, K in terms of $g_{\mu\nu}$, A_μ, β^μ, Λ_β, φ arranged in a derivative expansion, along with an associated N^μ, Δ such that Eq. (2.16) is satisfied. However, these $T^{\mu\nu}$, J^μ, K will not be the constitutive relations of a superfluid, as they will also have information about the surroundings. The true constitutive relations will be gained

[4] $K = 0$ should be thought of as a placeholder for the Josephson junction condition $u^\mu \xi_\mu = \mu + \mathcal{O}(\partial)$, which provides dynamics for the U(1) phase φ in the conventional treatment of superfluids. At the moment, however, we will allow for an arbitrary K treating it as yet another 'current' besides $T^{\mu\nu}$, J^μ in the theory, and will later establish that the second law of thermodynamics forces K to take the Josephson form.

by considering those solutions to Eq. (2.16) which do not involve a chosen set of terms that can be eliminated using the conservation equations Eq. (2.13) and the φ equation of motion Eq. (2.12).

Josephson equation: In the study of superfluids, the U(1) phase φ is generally taken to be order -1 in the derivative expansion, while its covariant derivative ξ_μ is taken to be order 0. This is because the true dynamical degrees of freedom are encoded in the fluctuations of φ along the U(1) circle, and not in φ itself. As a consequence, the $K\delta_{\mathcal{B}}\varphi$ term in the free energy conservation Eq. (2.16) can be order 0 when K has an order 0 term. This gives us a solution to Eq. (2.16) at zero derivative order, which was absent for ordinary fluids,

$$\begin{aligned} N^\mu, T^{\mu\nu}, J^\mu &= \mathcal{O}(\partial^0), \\ K &= -\alpha\delta_{\mathcal{B}}\varphi + \mathcal{O}(\partial), \\ \Delta &= \alpha(\delta_{\mathcal{B}}\varphi)^2 + \mathcal{O}(\partial), \end{aligned} \tag{2.18}$$

for some "transport coefficient" $\alpha \geq 0$. Note that the φ equation of motion at this order will read $K = -\alpha\delta_{\mathcal{B}}\varphi + \mathcal{O}(\partial) = 0$, implying,

$$\delta_{\mathcal{B}}\varphi = \frac{1}{T}(u^\mu\xi_\mu - \mu) = \mathcal{O}(\partial) \Rightarrow u^\mu\xi_\mu = \mu + \mathcal{O}(\partial). \tag{2.19}$$

This is the well known Josephson equation. This condition also ensures that Δ is at least $\mathcal{O}(\partial)$, avoiding "ideal superfluid dissipation".

C. Relativistic (super)fluids up to first order

In [27], author provides a complete classification and construction of the superfluid constitutive relations satisfying Eq. (2.16) up to arbitrarily high orders in derivative expansion. In this work, however, we are only concerned with superfluids up to first derivative order, which can be analyzed directly by brute force without involving the technicalities of [27]. Since these results have already been well explored in [12–14], in on-shell formalism, we only summarize the final results in the following. A detailed derivation in off-shell formalism can be found in Appendix A.

We find that the constitutive relations of a relativistic superfluid up to first derivative order are given as

$$\begin{aligned} T^{\mu\nu} &= (E+P)u^\mu u^\nu + Pg^{\mu\nu} + R_s\xi^\mu\xi^\nu + \mathcal{T}^{\mu\nu} + \mathcal{O}(\partial^2), \\ J^\mu &= Qu^\mu - R_s\xi^\mu + \mathcal{J}^\mu + \mathcal{O}(\partial^2), \\ J^\mu_S &= Su^\mu + \mathcal{S}^\mu + \mathcal{O}(\partial^2), \end{aligned} \tag{2.20}$$

where the energy density E, pressure P, superfluid density R_s, charge density Q and entropy density S are functions of the zero derivative scalars T, μ and $\mu_s = -\frac{1}{2}\xi^\mu\xi_\mu$. These functions are related to each other via the thermodynamic relations,

$$\begin{aligned} &\mathrm{d}P = S\mathrm{d}T + Q\mathrm{d}\mu + R_s\mathrm{d}\mu_s \quad \text{(Gibbs-Duhem)}, \\ &E + P = ST + Q\mu \quad \text{(Euler relation)}. \end{aligned} \tag{2.21}$$

On the other hand, $\mathcal{T}^{\mu\nu}$, $\mathcal{J}^\mu$ and $\mathcal{S}^\mu$ are first derivative corrections to the constitutive relations. They are characterized by 30 transport coefficients[5] which are functions of T, $\nu = \mu/T$, $\hat{\mu}_s = -\frac{1}{2}(g^{\mu\nu} + u^\mu u^\nu)\xi_\mu\xi_\nu$, and two constants C_1 and $C^{(4)}$. The constants C_1 and $C^{(4)}$ along with P and 4 transport coefficients,

$$\begin{aligned} \text{parity even}\,(2):&\ f_1, \quad f_2, \\ \text{parity odd}\,(2):&\ g_1, \quad g_2, \end{aligned} \tag{2.22}$$

totally determine the hydrostatic transport (part of the constitutive relations that survive at equilibrium). Nonhydrostatic, nondissipative transport (part that does not survive at equilibrium but doesn't contribute to $\Delta \geq 0$ either) is given by 11 transport coefficients,

$$\begin{aligned} \text{parity even}\,(7):&\ [\beta_{[ij]}]_{4\times4} \quad \text{(antisymmetric)}, \\ &\ [\kappa_{[ij]}]_{2\times2} \quad \text{(antisymmetric)}, \\ \text{parity odd}\,(4):&\ [\tilde{\kappa}_{(ij)}]_{2\times2} \quad \text{(symmetric)}, \quad \tilde{\eta}. \end{aligned} \tag{2.23}$$

Finally the entire dissipative transport is given by 15 transport coefficients ($\beta_{44} = \alpha/T$),

$$\begin{aligned} \text{parity even}\,(14):&\ [\beta_{(ij)}]_{4\times4} \quad \text{(symmetric)}, \\ &\ [\kappa_{(ij)}]_{2\times2} \quad \text{(symmetric)}, \quad \eta, \\ \text{parity odd}\,(1):&\ [\tilde{\kappa}_{[ij]}]_{2\times2} \quad \text{(antisymmetric)}. \end{aligned} \tag{2.24}$$

These dissipative transport coefficients follow a set of inequalities,

$$[\beta_{(ij)}]_{4\times4}, \qquad [\kappa'_{(ij)}]_{2\times2}, \qquad \eta \geq 0, \tag{2.25}$$

where

$$\kappa'_{ij} = \begin{pmatrix} \kappa'_{11} & \kappa'_{12} \\ \kappa'_{21} & \kappa'_{22} \end{pmatrix} = \begin{pmatrix} \kappa_{11} & \kappa_{12} \\ \kappa_{21} & \kappa_{22} + 2\hat{\mu}_s\frac{\tilde{\kappa}_{[12]}}{\kappa_{11}} \end{pmatrix}, \tag{2.26}$$

and a "non-negative matrix" implies all its eigenvalues are non-negative.

Out of these for an ordinary relativistic fluid, shear viscosity η, bulk viscosity $\zeta = \beta_{11}$, conductivity $\kappa = \kappa_{22}$ and the constants C_1, $C^{(4)}$ are present. In addition g_1 and g_2 are forced to be constants, while all the remaining transport coefficients zero.

[5] Our parity-odd counting is only valid in $3+1$ dimensions.

Defining the differentials of f_i and g_i as

$$\begin{aligned} \mathrm{d}f_i &= \frac{\alpha_{E,i}}{T}\mathrm{d}T + T\alpha_{Q,i}\mathrm{d}\nu + \left(\alpha_{R_s,i} - \frac{f_i}{2\hat{\mu}_s}\right)\mathrm{d}\hat{\mu}_s, \qquad \alpha_{E,i} + f_i = \alpha_{S,i}T + \alpha_{Q,i}\mu, \\ \mathrm{d}g_i &= \frac{\tilde{\alpha}_{E,i}}{T}\mathrm{d}T + T\tilde{\alpha}_{Q,i}\mathrm{d}\nu + \left(\tilde{\alpha}_{R_s,i} - \frac{g_i}{2\hat{\mu}_s}\right)\mathrm{d}\hat{\mu}_s, \qquad \tilde{\alpha}_{E,i} + g_i = \tilde{\alpha}_{S,i}T + \tilde{\alpha}_{Q,i}\mu, \end{aligned} \tag{2.27}$$

the first derivative corrections to the constitutive relations are given as: the energy-momentum tensor,

$$\begin{aligned} \mathcal{T}^{\mu\nu} &= u^\mu u^\nu \left[\sum_{i=1}^{2}\alpha_{E,i}S_{e,i} + \sum_{i=1}^{2}\tilde{\alpha}_{E,i}\tilde{S}_{e,i} - \frac{1}{T}\nabla_\sigma(Tf_1\zeta^\sigma) + \epsilon^{\alpha\rho\sigma\tau}u_\alpha\nabla_\rho(Tg_1u_\sigma\zeta_\tau)\right] \\ &\quad + 2u^{(\mu}\zeta^{\nu)}\left[\sum_{i=1}^{2} f_i S_{4+i} - (u^\rho\xi_\rho)\left(\sum_{i=1}^{2}\alpha_{R_s,i}S_{e,i} + \sum_{i=1}^{2}\tilde{\alpha}_{R_s,i}\tilde{S}_{e,i}\right) + \frac{1}{2\hat{\mu}_s}\epsilon^{\alpha\rho\sigma\tau}\zeta_\alpha\nabla_\rho(Tg_1u_\sigma\zeta_\tau)\right] \\ &\quad + \zeta^\mu\zeta^\nu\left[\sum_{i=1}^{2}\alpha_{R_s,i}S_{e,i} + \sum_{i=1}^{2}\left(\tilde{\alpha}_{R_s,i} - \frac{g_i}{2\hat{\mu}_s}\right)\tilde{S}_{e,i} - \sum_{i=1}^{4}\beta_{2i}S_i\right] \\ &\quad + 2u^{(\mu}\left[(\xi^\sigma u_\sigma)\sum_{i=1}^{2} f_i V_{e,i}^{\nu)} - \sum_{i=1}^{2} g_i\tilde{V}_{e,2+i}^{\nu)} - \tilde{P}_\alpha^{\nu)}\epsilon^{\alpha\rho\sigma\tau}\nabla_\rho(Tg_1u_\sigma\zeta_\tau) + 2C_1T^3\omega^{\nu)} + C^{(4)}\mu^2(3M^{\nu)} + 2\mu\omega^{\nu)})\right] \\ &\quad - 2\zeta^{(\mu}\left[\sum_{i=1}^{2} f_i V_{e,i}^{\nu)} + \sum_{i=1}^{2}\kappa_{1i}V_i^{\nu)} + \sum_{i=1}^{2}\tilde{\kappa}_{1i}\tilde{V}_i^{\nu)}\right] + \tilde{P}^{\mu\nu}\left[\sum_{i=1}^{2} f_i S_{e,i} - \sum_{i=1}^{4}\beta_{1i}S_i\right] - \eta\sigma^{\mu\nu} - \tilde{\eta}\tilde{\sigma}^{\mu\nu}, \end{aligned} \tag{2.28}$$

the charge current,

$$\begin{aligned} \mathcal{J}^\mu &= u^\mu\left[\sum_{i=1}^{2}\alpha_{Q,i}S_{e,i} + \sum_{i=1}^{2}\tilde{\alpha}_{Q,i}\tilde{S}_{e,i} - \frac{1}{T}\nabla_\nu(Tf_2\zeta^\nu) + \epsilon^{\alpha\nu\rho\sigma}u_\alpha\nabla_\nu(Tg_2u_\rho\zeta_\sigma)\right] \\ &\quad - \zeta^\mu\left[\sum_{i=1}^{2}\alpha_{R_s,i}S_{e,i} + \sum_{i=1}^{2}\tilde{\alpha}_{R_s,i}\tilde{S}_{e,i} + \sum_{i=1}^{4}\beta_{3i}S_i - \frac{1}{2\hat{\mu}_s}\epsilon^{\alpha\nu\rho\sigma}\zeta_\alpha\nabla_\nu(Tg_2u_\rho\zeta_\sigma)\right] \\ &\quad + \sum_{i=1}^{2} f_i V_{e,i}^\mu + \sum_{i=1}^{2} g_i\tilde{V}_{e,i}^\mu - \sum_{i=1}^{2}\kappa_{2i}V_i^\mu - \sum_{i=1}^{2}\tilde{\kappa}_{2i}\tilde{V}_i^\mu - \tilde{P}_\alpha^\mu\epsilon^{\alpha\nu\rho\sigma}\nabla_\nu(Tg_2u_\rho\zeta_\sigma) + 3\mu C^{(4)}(2M^\mu + \mu\omega^\mu), \end{aligned} \tag{2.29}$$

and the entropy current,

$$\begin{aligned} \mathcal{S}^\mu &= g_1\frac{1}{T}\epsilon^{\mu\nu\rho\sigma}u_\nu\zeta_\rho\partial_\sigma T + g_2T\epsilon^{\mu\nu\rho\sigma}u_\nu\xi_\rho\partial_\sigma\nu + 3C_1T^2\omega^\mu \\ &\quad + u^\mu\left[\sum_{i=1}^{2}\alpha_{S,i}S_{e,i} + \sum_{i=1}^{2}\tilde{\alpha}_{S,i}\tilde{S}_{e,i} - \frac{1}{T^2}\nabla_\sigma(Tf_1\zeta^\sigma) + \frac{\mu}{T^2}\nabla_\nu(Tf_2\zeta^\nu) + \frac{1}{T}\epsilon^{\alpha\nu\rho\sigma}u_\alpha\nabla_\nu(Tg_1u_\rho\zeta_\sigma) - \frac{\mu}{T}\epsilon^{\alpha\rho\sigma\tau}u_\alpha\nabla_\nu(Tg_2u_\rho\zeta_\sigma)\right] \\ &\quad + \frac{1}{T}\zeta^\mu\left[\sum_{i=1}^{4}\mu\beta_{3i}S_i + \frac{1}{2\hat{\mu}_s}\epsilon^{\alpha\rho\sigma\tau}\zeta_\alpha\nabla_\rho(Tg_1u_\sigma\zeta_\tau) - \frac{\mu}{2\hat{\mu}_s}\zeta_\alpha\epsilon^{\alpha\nu\rho\sigma}\nabla_\nu(Tg_2u_\rho\zeta_\sigma)\right] \\ &\quad + \frac{\mu}{T}\sum_{i=1}^{2}\kappa_{2i}V_i^\mu + \frac{\mu}{T}\sum_{i=1}^{2}\tilde{\kappa}_{2i}\tilde{V}_i^\mu - \frac{1}{T}\tilde{P}_\alpha^\mu\epsilon^{\alpha\nu\rho\sigma}\nabla_\nu(Tg_1u_\rho\zeta_\sigma) + \frac{\mu}{T}\tilde{P}_\alpha^\mu\epsilon^{\alpha\nu\rho\sigma}\nabla_\nu(Tg_2u_\rho\zeta_\sigma). \end{aligned} \tag{2.30}$$

Here $\zeta^\mu = (g^{\mu\nu} + u^\mu u^\nu)\zeta_\nu$, $\tilde{P}^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu - \frac{1}{\zeta^\sigma\zeta_\sigma}\zeta^\mu\zeta^\nu$, $\omega^\mu = \epsilon^{\mu\nu\rho\sigma}u_\nu\partial_\rho u_\sigma$ and $M^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}u_\nu F_{\rho\sigma}$. Various first derivative order data appearing here have been defined in Table II. Finally, corrections to the Josephson equation ($K = 0$) coming from the first-order superfluid transport are given as (here $\beta_{44} = \alpha/T$),

$$u^\mu\xi_\mu = \mu + \frac{1}{\beta_{44}}\nabla_\mu(R_s\xi^\mu) - \sum_{i=1}^{3}\frac{\beta_{4i}}{\beta_{44}}S_i + \frac{1}{\beta_{44}}\nabla_\mu\left(\zeta^\mu\sum_{i=1}^{2}\alpha_{R_s,i}S_{e,i} + \zeta^\mu\sum_{i=1}^{2}\tilde{\alpha}_{R_s,i}\tilde{S}_{e,i} - \sum_{i=1}^{2} f_i V_{e,i}^\mu - \sum_{i=1}^{2} g_i\tilde{V}_{e,i}^\mu\right) + \mathcal{O}(\partial^2), \tag{2.31}$$

which can be seen as determining $u^\mu\xi_\mu$ in terms of the other superfluid variables. Note that though this equation contains second-order terms, it is only correct up to the first order in derivatives, and will admit further corrections coming from higher-order superfluid transport.

TABLE II. Independent first-order data for relativistic superfluids. We have not enlisted, neither would we need, all the independent data surviving at equilibrium.

Vanishing at equilibrium—On-shell independent		
S_1	$\frac{T}{2}\tilde{P}^{\mu\nu}\delta_{\mathscr{B}}g_{\mu\nu}$	$\tilde{P}^{\mu\nu}\nabla_\mu u_\nu$
S_2	$\frac{T}{2}\zeta^\mu\zeta^\nu\delta_{\mathscr{B}}g_{\mu\nu}$	$\zeta^\mu\zeta^\nu\nabla_\mu u_\nu$
S_3	$T\zeta^\mu\delta_{\mathscr{B}}A_\mu$	$\zeta^\mu(T\nabla_\mu\nu + u^\nu F_{\nu\mu})$
S_4	$T\delta_{\mathscr{B}}\varphi$	$u^\mu\xi_\mu - \mu$
V_1^μ	$T\tilde{P}^{\mu\nu}\zeta^\rho\delta_{\mathscr{B}}g_{\nu\rho}$	$2\tilde{P}^{\mu\nu}\zeta^\rho\nabla_{(\nu}u_{\rho)}$
V_2^μ	$T\tilde{P}^{\mu\nu}\delta_{\mathscr{B}}A_\mu$	$\tilde{P}^{\mu\nu}(T\nabla_\nu\nu + u^\sigma F_{\sigma\nu})$
$\sigma^{\mu\nu}$	$\frac{T}{2}\tilde{P}^{\rho\langle\mu}\tilde{P}^{\nu\rangle\sigma}\delta_{\mathscr{B}}g_{\rho\sigma}$	$\tilde{P}^{\mu\rho}\tilde{P}^{\nu\sigma}(\nabla_{(\rho}u_{\sigma)} - \frac{1}{d-2}\tilde{P}_{\rho\sigma}S_1)$
$\tilde{V}_1^\mu$	$\epsilon^{\mu\nu\rho\sigma}u_\nu\zeta_\rho V_{1,\sigma}$	
$\tilde{V}_2^\mu$	$\epsilon^{\mu\nu\rho\sigma}u_\nu\zeta_\rho V_{2,\sigma}$	
$\tilde{\sigma}^{\mu\nu}$	$\epsilon^{(\mu\|\rho\sigma\tau}u_\rho\zeta_\sigma\sigma_\tau^{\nu)}$	
Vanishing at equilibrium—On-shell dependent		
S_5	$\frac{T}{2}u^\mu u^\nu\delta_{\mathscr{B}}g_{\mu\nu}$	$\frac{1}{T}u^\mu\nabla_\mu T$
S_6	$Tu^\mu\delta_{\mathscr{B}}A_\mu$	$Tu^\mu\nabla_\mu\nu$
S_7	$T\zeta^\mu u^\nu\delta_{\mathscr{B}}g_{\mu\nu}$	$\zeta^\nu(\frac{1}{T}\nabla_\nu T + u^\sigma\nabla_\sigma u_\nu)$
V_3^μ	$T\tilde{P}^{\mu\nu}u^\rho\delta_{\mathscr{B}}g_{\nu\rho}$	$\tilde{P}^{\mu\nu}(\frac{1}{T}\nabla_\nu T + u^\sigma\nabla_\sigma u_\nu)$
$\tilde{V}_3^\mu$	$\epsilon^{\mu\nu\rho\sigma}u_\nu\zeta_\rho V_{3,\sigma}$	
Surviving at equilibrium		
$S_{e,1}$	$\frac{1}{T}\zeta^\mu\partial_\mu T$	
$S_{e,2}$	$T\zeta^\mu\partial_\mu\nu$	
$S_{e,3}$	$\zeta^\mu\partial_\mu\hat{\mu}_s$	
$S_{e,4}$	$\nabla_\mu\zeta^\mu$	
$V_{e,1}^\mu$	$\frac{1}{T}\tilde{P}^{\mu\nu}\partial_\nu T$	
$V_{e,2}^\mu$	$T\tilde{P}^{\mu\nu}\partial_\nu\nu$	
$\tilde{S}_{e,1}$	$T\epsilon^{\mu\nu\rho\sigma}\zeta_\mu u_\nu\partial_\rho u_\sigma$	
$\tilde{S}_{e,2}$	$\frac{1}{2}T\epsilon^{\mu\nu\rho\sigma}\zeta_\mu u_\nu F_{\rho\sigma}$	
$\tilde{V}_{e,1}^\mu$	$T\tilde{P}_\tau^\mu\epsilon^{\tau\nu\rho\sigma}u_\nu\partial_\rho u_\sigma$	
$\tilde{V}_{e,2}^\mu$	$\frac{1}{2}T\tilde{P}_\tau^\mu\epsilon^{\tau\nu\rho\sigma}u_\nu F_{\rho\sigma}$	
$\tilde{V}_{e,3}^\mu$	$T\tilde{P}_\tau^\mu\epsilon^{\tau\nu\rho\sigma}\xi_\nu\partial_\rho u_\sigma$	
$\tilde{V}_{e,4}^\mu$	$\frac{1}{2}T\tilde{P}_\tau^\mu\epsilon^{\tau\nu\rho\sigma}\xi_\nu F_{\rho\sigma}$	
$\vdots$	$\vdots$	

It should be noted that these results are presented in a particular hydrodynamic frame (gained by aligning u^μ, T, μ along β^μ, Λ_β) and in a "natural" choice of basis for the independent data. They can be transformed to any other preferred hydrodynamic frame or basis by a straight forward substitution.

In deriving these constitutive relations, we have only used the second law of thermodynamics. To compare these results with the existing literature [12–14], one might need to further filter these results with requirements like microscopic reversibility (Onsager relations), time reversal invariance and CPT invariance. For example, Onsager relations are known to turn off 7 parity-even nondissipative coefficients $[\beta_{[ij]}]_{4\times 4}$, $[\kappa_{[ij]}]_{2\times 2}$ and the only parity-odd dissipative coefficient $[\tilde{\kappa}_{[ij]}]_{2\times 2}$ [12]. To avoid confusion, also note that there is a coefficient f_3 appearing in Eq. (A11) which we removed by using the φ equation of motion (or equivalently, by redefining φ). This coefficient has been included in the counting of independent transport coefficients in [13].

III. NULL SUPERFLUIDS

In [17] we proposed "null fluids" as a new viewpoint of Galilean fluids. In this section, we will further extend this formalism to include Galilean superfluids. The main benefit of working with "null (super)fluids" is that it is a "relativistic embedding" of Galilean (super)fluids into one higher dimension and enables us to directly use the existing relativistic machinery to read out the respective Galilean results. In this sense, our in-depth review of relativistic superfluids in the previous section will be vital for our discussion of null/Galilean superfluids. Later in Sec. IV, we will translate our null superfluid results to the better known Newton-Cartan and conventional noncovariant notations.

A. Null backgrounds and null superfluids

Let us quickly recap null backgrounds [17,18], which are a natural "embedding" of Galilean (Newton-Cartan) backgrounds into a relativistic spacetime of one higher dimension. Consider a $(d+1)$-dimensional manifold $\mathcal{M}_{(d+1)}$ equipped with a metric g_{MN} and a U(1) gauge field A_M. Infinitesimal diffeomorphisms and gauge transformation with parameters $\mathcal{X} = \{\chi^M, \Lambda_\chi\}$ respectively, act on these background fields as

$$\delta_{\mathcal{X}}g_{MN} = \nabla_M\chi_N + \nabla_N\chi_M,$$
$$\delta_{\mathcal{X}}A_M = \partial_M(\Lambda_\chi + \chi^N A_N) + \chi^N F_{NM}. \tag{3.1}$$

The characteristic feature of a null background is the existence of a compatible null isometry $\mathcal{V} = \{V^M, \Lambda_V\}$ which satisfies: $V^M V_M = 0$, $\nabla_M V^N = 0$, $V^M A_M + \Lambda_V = -1$[6] and,

$$\delta_{\mathcal{V}}g_{MN} = \nabla_M V_N + \nabla_N V_M = 0,$$
$$\delta_{\mathcal{V}}A_M = \partial_M(\Lambda_V + V^N A_N) + V^N F_{NM} = V^N F_{NM} = 0. \tag{3.2}$$

Since we will be interested in studying superfluids on this background, we introduce a preferred U(1) phase φ which transforms under diffeomorphisms and infinitesimal gauge transformations as $\delta_{\mathcal{X}}\varphi = \chi^M\partial_M\varphi - \Lambda_\chi$. The covariant derivative of φ is known as the *superfluid velocity*,

[6]This condition can be thought of as fixing a component of the $(d+1)$-dimensional gauge field A_M, leaving it with only d independent components mapping bijectively to the d-dimensional Galilean gauge field. As opposed to the null backgrounds defined in [17] where we set $V^M A_M + \Lambda_V = 0$, for superfluids we realize that it is more suitable to fix $V^M A_M + \Lambda_V = -1$ instead.

$$\xi_M = \partial_M \varphi + A_M. \tag{3.3}$$

We require φ to respect the null isometry $\mathcal{V}$, i.e. $\delta_{\mathcal{V}}\varphi = V^M\partial_M\varphi - \Lambda_V = V^M\xi_M + 1 = 0$, which implies $V^M\xi_M = -1$. The remainder of the story is exactly same as the relativistic case: any theory coupled to a null background has an energy-momentum tensor T^{MN} and a charge current J^M in its spectrum. The respective conservation laws are given as

$$\nabla_M T^{MN} = F^{NR}J_R + \mathrm{T}^{N\perp}_{\mathrm{H}} + \xi^M K,$$
$$\nabla_M J^M = \mathrm{J}^{\perp}_{\mathrm{H}} - K, \tag{3.4}$$

where

$$K = 0, \tag{3.5}$$

is the φ equation of motion. Since Eqs. (3.4) and (3.5) are $(d+3)$ equations in $(d+1)$ dimensions, they can provide dynamics for a superfluid described by an arbitrary set of $(d+2)$ variables in addition to the phase φ. We choose these to be a normalized null fluid velocity u^M (with $u^M V_M = -1$, $u^M u_M = 0$), a temperature T, a mass chemical potential μ_n, and a chemical potential μ, known as the hydrodynamic fields. A null superfluid hence is completely characterized by gauge-invariant expressions of T^{MN}, J^M, K in terms of g_{MN}, A_M, u^M, T, μ_n, μ and ξ_M, known as the null superfluid constitutive relations. The near equilibrium assumption allows us to arrange these constitutive relations as a perturbative expansion in derivatives (known as the derivative or gradiant expansion).

Same as the relativistic case, null superfluid is also required to satisfy a version of the second law of thermodynamics. It states that there must exist an entropy current J^M_S whose divergence is positive semidefinite everywhere, i.e.,

$$\nabla_M J^M_S \geq 0, \tag{3.6}$$

as long as the superfluid is thermodynamically isolated (i.e. conservation laws Eq. (3.4) are satisfied), irrespective of K being zero. The job of null superfluid dynamics now is to find the most general constitutive relations T^{MN}, J^M, K and an associated J^M_S, Δ order by order in derivative expansion, such that Eq. (3.6) is satisfied for thermodynamically isolated fluids. Owing to our previous experiences with the second law, however, we switch to the off-shell formalism in the next subsection for simplicity.

B. Off-shell formalism for null (super)fluids

We couple the fluid to an external momentum P^M_{ext} and charge Q_{ext} source, so that the conservation laws are no longer satisfied. Having done that, the second law Eq. (3.6) will be modified with an arbitrary combination of the conservation laws to get,

$$\nabla_M J^M_S + \beta_N(\nabla_M T^{MN} - F^{NR}J_R - \mathrm{T}^{N\perp}_{\mathrm{H}} - \xi^M K)$$
$$+ (\Lambda_\beta + A_M\beta^M)(\nabla_M J^M - \mathrm{J}^{\perp}_{\mathrm{H}} + K) = \Delta \geq 0, \tag{3.7}$$

where $\mathcal{B} = \{\beta^M, \Lambda_\beta\}$ are some arbitrary fields. Recall that the hydrodynamic fields u^M, T, μ_n, μ were some arbitrary $(d+2)$ fields chosen to describe the fluid. Like in any field theory, they are permitted to admit an arbitrary redefinition among themselves without changing the physics. This huge amount of freedom can be fixed by explicitly choosing,

$$u^M = -\frac{\beta^M}{V_M\beta^M} + \frac{\beta^R\beta_R V^M}{2(V_N\beta^N)^2}, \qquad T = -\frac{1}{V_M\beta^M},$$
$$\mu_n = \frac{\beta^M U_M}{2(V_N\beta^N)^2}, \qquad \mu = -\frac{\Lambda_\beta + A_M\beta^M}{V_N\beta^N}. \tag{3.8}$$

or conversely,

$$\beta^M = \frac{1}{T}(u^M - \mu_n V^M), \qquad \Lambda_\beta = \frac{\mu}{T} - A_M u^M. \tag{3.9}$$

We define a free energy current,

$$-\frac{G^M}{T} = N^M = J^M_S + T^{MN}\beta_N + (\Lambda_\beta + \beta^N A_N)J^M,$$
$$-\frac{\mathrm{G}^M_{\mathrm{H}}}{T} = \mathrm{N}^{\perp}_{\mathrm{H}} = \beta_M \mathrm{T}^{M\perp}_{\mathrm{H}} + (\Lambda_\beta + \beta^M A_M)\mathrm{J}^{\perp}_{\mathrm{H}}, \tag{3.10}$$

which turns the off-shell second law in Eq. (3.7) to a free energy conservation equation,

$$\nabla_M N^M - \mathrm{N}^{\perp}_{\mathrm{H}} = \frac{1}{2}T^{MN}\delta_{\mathcal{B}}g_{MN} + J^M\delta_{\mathcal{B}}A_M + K\delta_{\mathcal{B}}\varphi + \Delta,$$
$$\Delta \geq 0. \tag{3.11}$$

Now similar to our analysis of relativistic superfluids, we will try to find the most generic T^{MN}, J^M, K in terms of g_{MN}, A_M, β^M, Λ_β, φ which solves this equation for some N^M, Δ. Again, however, these expressions will be shy of being the null superfluid constitutive relations because of their dependence on the external sources P^M_{ext}, Q_{ext}. To fix this, we will only consider the expressions for T^{MN}, J^M, K which are independent of certain data that can be eliminated using the conservation laws.

Josephson equation: Following our discussion of relativistic superfluids, Eq. (3.11) has a zero derivative order solution,

$$N^M, T^{MN}, J^M = \mathcal{O}(\partial^0),$$
$$K = -\alpha\delta_{\mathcal{B}}\varphi + \mathcal{O}(\partial), \qquad \Delta = \alpha(\delta_{\mathcal{B}}\varphi)^2 + \mathcal{O}(\partial), \tag{3.12}$$

for some "transport coefficient" $\alpha \geq 0$. The φ equation of motion $K = 0$ then implies the Josephson equation for null superfluids,

$$\delta_{\mathcal{B}}\varphi = \frac{1}{T}(u^M\xi_M + \mu_n - \mu) = \mathcal{O}(\partial)$$
$$\Rightarrow u^M\xi_M = \mu - \mu_n + \mathcal{O}(\partial). \tag{3.13}$$

This condition also ensures that Δ is at least $\mathcal{O}(\partial)$, avoiding "ideal superfluid dissipation". Note that this equation determines $\delta_{\mathcal{B}}\varphi$ in terms of first- and higher-order data; therefore, it would be beneficial from here onward to think of $\delta_{\mathcal{B}}\varphi$ as order-one data in derivative expansion.

C. Ideal null superfluids

Let us now move on to the ideal null superfluids, i.e. null superfluid constitutive relations that satisfy the free energy conservation Eq. (3.11) at first derivative order. At ideal order, the most generic tensorial form of various quantities appearing in Eq. (3.11) can be written as

$$\begin{aligned}
T^{MN} &= R_n u^M u^N + 2Eu^{(M}V^{N)} + PP^{MN} + R_s\xi^M\xi^N \\
&\quad + 2\lambda_1\xi^{(M}V^{N)} + 2\lambda_2\xi^{(M}u^{N)} + R_v V^M V^M + \mathcal{O}(\partial), \\
J^M &= Qu^M + Q_s\xi^M + Q_v V^M + \mathcal{O}(\partial), \\
K &= -\alpha\delta_{\mathcal{B}}\varphi + K_{\text{ideal}} + \mathcal{O}(\partial), \\
N^M &= Nu^M + N_s\xi^M + N_v V^M + \mathcal{O}(\partial), \\
\Delta &= (\alpha\delta_{\mathcal{B}}\varphi)^2 + \Delta_{\text{ideal}} + \mathcal{O}(\partial^2),
\end{aligned} \tag{3.14}$$

where R_n, E, P, R_s, λ_1, λ_2, Q, Q_s, K_{ideal}, N, N_s are functions of T, μ, μ_n and $\mu_s \equiv -\frac{1}{2}\xi^M\xi_M$. We have omitted the only other possible scalar $\delta_{\mathcal{B}}\varphi$ in the functional dependence, because using the φ equation of motion we know that it is no longer an independent quantity. The coefficients R_v, Q_v, N_v do not contain any physical information, because their contribution to the conservation laws trivially vanish owing to $\mathcal{V}$ being an isometry. Plugging Eq. (3.14) in Eq. (3.11) we can find,

$$\begin{aligned}
&(Q_s + R_s)\xi^M\left(\nabla_M\nu + \frac{1}{T}u^N F_{NM}\right) + \frac{\lambda_1}{T^2}\xi^M\nabla_M T + \lambda_2\xi^N(\nabla_N\nu_n + u^M\nabla_M U_N) \\
&\nabla_M\left(\left(\frac{P}{T} - N\right)u^N\right) + \frac{1}{T}u^\mu(\nabla_\mu E - T\nabla_M S - \mu_n\nabla_M R_n - \mu\nabla_M Q + R_s\nabla_N\mu_s) + \nabla_M((\delta_{\mathcal{B}}\varphi R_s - N_s)\xi^M) \\
&\quad + (K_{\text{ideal}} - \nabla_M(R_s\xi^M))\delta_{\mathcal{B}}\varphi + \Delta_{\text{ideal}} = 0,
\end{aligned} \tag{3.15}$$

where we have defined S through the "Euler equation",

$$E + P = ST + Q\mu + R_n\mu_n. \tag{3.16}$$

Equation (3.15) will imply a set of relations among various coefficients,

$$\begin{aligned}
&Q_s = -R_s, \qquad \lambda_1 = \lambda_2 = 0, \qquad N = \frac{P}{T}, \\
&N_s = \delta_{\mathcal{B}}\varphi R_s, \qquad K_{\text{ideal}} = \nabla_M(R_s\xi^M), \qquad \Delta_{\text{ideal}} = 0,
\end{aligned} \tag{3.17}$$

and the "first law of thermodynamics,"

$$\mathrm{d}E = T\mathrm{d}S + \mu\mathrm{d}Q + \mu_n\mathrm{d}R_n - R_s\mathrm{d}\mu_s, \tag{3.18}$$

giving physical meaning to the quantities we have introduced in Eq. (3.14). Finally, we have the full set of null superfluid constitutive relations up to ideal order satisfying the second law,

$$\begin{aligned}
T^{MN} &= R_n u^M u^N + 2Eu^{(M}V^{N)} + PP^{MN} + R_s\xi^M\xi^N \\
&\quad + R_v V^M V^N + \mathcal{O}(\partial), \\
J^M &= Qu^M - R_s\xi^M + Q_v V^M + \mathcal{O}(\partial), \\
K &= -\alpha\delta_{\mathcal{B}}\varphi + \nabla_M(R_s\xi^M) + \mathcal{O}(\partial), \\
N^M &= \frac{P}{T}u^M + \delta_{\mathcal{B}}\varphi R_s\xi^M + N_v V^M + \mathcal{O}(\partial), \\
J_S^M &= N^M - \frac{1}{T}(T^{MN}u_N - \mu_n T^{MN}V_N + \mu J^M) \\
&= Su^M + S_v V^M + \mathcal{O}(\partial).
\end{aligned} \tag{3.19}$$

Here $S_v = N_v + \frac{1}{T}(R_v - \mu_n E - \mu Q_v)$, which again doesn't contain any physical information. These are the ideal null superfluid constitutive relations. Note that we have included first-order terms in K, N^M which can be ignored when talking about the ideal order, but are required for internal consistency with Eq. (3.11). The φ equation of motion $K = 0$ will imply,

$$\alpha\delta_{\mathcal{B}}\varphi = \nabla_M(R_s\xi^M) + \mathcal{O}(\partial)$$
$$\Rightarrow u^M\xi_M = \mu - \mu_n + \frac{T}{\alpha}\nabla_M(R_s\xi^M) + \mathcal{O}(\partial), \tag{3.20}$$

TABLE III. Independent first-order data for null superfluids. We have not enlisted, neither would we need, all the independent data surviving at equilibrium.

Vanishing at equilibrium—On-shell independent		
S_1	$\frac{T}{2}\tilde{P}^{MN}\delta_{\mathcal{B}}g_{MN}$	$\tilde{P}^{MN}\nabla_M u_N$
S_2, $S_{e,1}$[7]	$TV^M\zeta^N\delta_{\mathcal{B}}g_{MN}$	$\frac{1}{T}\zeta^M\nabla_M T$
S_3	$\frac{T}{2}\zeta^M\zeta^N\delta_{\mathcal{B}}g_{MN}$	$\zeta^M\zeta^N\nabla_M u_N$
S_4	$T\zeta^M\delta_{\mathcal{B}}A_M$	$\zeta^M(T\nabla_M\nu + u^N F_{NM})$
S_5	$T\delta_{\mathcal{B}}\varphi$	$u^M\xi_M + \mu_n - \mu$
V_1^M, $V_{e,1}^M$	$T\tilde{P}^{MR}V^N\delta_{\mathcal{B}}g_{RN}$	$\frac{1}{T}\tilde{P}^{MN}\nabla_N T$
V_2^M	$T\tilde{P}^{MR}\zeta^N\delta_{\mathcal{B}}g_{RN}$	$2\tilde{P}^{MR}\zeta^N\nabla_{(R}u_{N)}$
V_3^M	$T\tilde{P}^{MN}\delta_{\mathcal{B}}A_N$	$\tilde{P}^{MN}(T\nabla_N\nu + u^R F_{RN})$
σ^{MN}	$\frac{T}{2}\tilde{P}^{R\langle M}\tilde{P}^{N\rangle S}\delta_{\mathcal{B}}g_{RS}$	$\tilde{P}^{MR}\tilde{P}^{NS}(\nabla_{(R}u_{S)} - \frac{\tilde{P}_{RS}}{d-1}S_1)$
$\tilde{V}_1^M$	$\epsilon^{MNRST}V_N u_R\zeta_S V_{1,T}$	
$\tilde{V}_2^M$	$\epsilon^{MNRST}V_N u_R\zeta_S V_{2,T}$	
$\tilde{V}_3^M$	$\epsilon^{MNRST}V_N u_R\zeta_S V_{3,T}$	
$\tilde{\sigma}^{MN}$	$\epsilon^{(M\vert RSTP}V_R u_S\zeta_T\sigma_P{}^{N)}$	
Vanishing at equilibrium—On-shell dependent		
S_6	$Tu^M V^N\delta_{\mathcal{B}}g_{MN}$	$\frac{1}{T}u^M\nabla_M T$
S_7	$Tu^M\delta_{\mathcal{B}}A_M$	$Tu^M\nabla_M\nu$
S_8	$\frac{T}{2}u^M u^N\delta_{\mathcal{B}}g_{MN}$	$Tu^M\nabla_M\nu_n$
S_9	$Tu^M\zeta^N\delta_{\mathcal{B}}g_{MN}$	$\zeta^M(T\nabla_M\nu_n + u^N\nabla_N u_M)$
V_4^M	$T\tilde{P}^{MR}u^N\delta_{\mathcal{B}}g_{RN}$	$\tilde{P}^{MN}(T\nabla_N\nu_n + u^R\nabla_R u_N)$
$\tilde{V}_4^M$	$\epsilon^{MNRST}V_N u_R\zeta_S V_{4,T}$	
Surviving at equilibrium		
$S_{e,2}$	$T\zeta^M\partial_M\nu$	
$S_{e,3}$	$T\zeta^M\partial_M\nu_n$	
$V_{e,2}^M$	$T\tilde{P}^{MN}\partial_N\nu$	
$V_{e,3}^M$	$T\tilde{P}^{MN}\partial_N\nu_n$	
$\tilde{S}_{e,1}$	$T\epsilon^{MNRST}\zeta_M V_N u_R\partial_S u_T$	
$\tilde{S}_{e,2}$	$\frac{1}{2}T\epsilon^{MNRST}\zeta_M V_N u_R F_{ST}$	
$\tilde{V}_{e,1}^M$	$T\tilde{P}_K^M\epsilon^{KNRST}V_N u_R\partial_S u_T$	
$\tilde{V}_{e,2}^M$	$\frac{1}{2}T\tilde{P}_K^M\epsilon^{KNRST}V_N u_R F_{ST}$	
$\tilde{V}_{e,3}^M$	$T\tilde{P}_K^M\epsilon^{KNRST}\xi_N u_R\partial_S u_T$	
$\tilde{V}_{e,4}^M$	$\frac{1}{2}T\tilde{P}_K^M\epsilon^{KNRST}\xi_N u_R F_{ST}$	
$\vdots$	$\vdots$	

which is a first-order correction to the Josephson equation. Note, however, that this equation can admit further one derivative corrections due to the first-order constitutive relations discussed in the next subsection; the correction mentioned here is only how the ideal null superfluid transport affects the Josephson equation. The conservation laws on the other hand are complete up to the first order in derivatives,

$$\frac{1}{\sqrt{-g}}\delta_{\mathcal{B}}(\sqrt{-g}(T(E+P)V_M + RTu_M)) + QT\delta_{\mathcal{B}}A_M = -\xi_M\alpha\delta_{\mathcal{B}}\varphi + \mathcal{O}(\partial^2),$$
$$\frac{1}{\sqrt{-g}}\delta_{\mathcal{B}}(\sqrt{-g}QT) = \alpha\delta_{\mathcal{B}}\varphi + \mathcal{O}(\partial^2). \tag{3.21}$$

These equations provide a set of relations between $\delta_{\mathcal{B}}\varphi$, $\delta_{\mathcal{B}}g_{MN}$ and $\delta_{\mathcal{B}}A_M$, which can be used to eliminate a vector $u^M\delta_{\mathcal{B}}g_{MN}$ and a scalar $u^M\delta_{\mathcal{B}}A_M$ (see Table III) from the first-order null constitutive relations. On the other hand, we choose to eliminate the scalar data $\nabla_M(R_s\xi^M)$ using the φ equation of motion.

D. First derivative corrections to null superfluids

Moving on to the one derivative null superfluids, let us schematically represent various quantities appearing in Eq. (3.11) up to the first order in derivatives as

$$\begin{aligned}
T^{MN} &= [R_n u^M u^N + 2Eu^{(M}V^{N)} + PP^{MN} + R_s\xi^M\xi^N \\
&\quad + R_v V^M V^N] + \mathcal{T}^{MN} + \mathcal{O}(\partial^2),\\
J^M &= [Qu^M - R_s\xi^M + Q_v V^M] + \mathcal{J}^M + \mathcal{O}(\partial^2),\\
K &= [-\alpha\delta_{\mathcal{B}}\varphi + \nabla_M(R_s\xi^M)] + \mathcal{K} + \mathcal{O}(\partial^2),\\
N^M &= \left[\frac{P}{T}u^M + \delta_{\mathcal{B}}\varphi R_s\xi^M + N_v V^M\right] + \mathcal{N}^M + \mathcal{O}(\partial^2),\\
\Delta &= \alpha(\delta_{\mathcal{B}}\varphi)^2 + \mathcal{D},
\end{aligned} \tag{3.22}$$

where the corrections $\mathcal{T}^{MN}$, $\mathcal{J}^M$, $\mathcal{K}$, $\mathcal{N}^M$, $\mathcal{D}$ have exactly one derivative in every term. Plugging these in the Eq. (3.11), we can get an equation among the corrections

$$\nabla_M\mathcal{N}^M - \mathrm{N}_{\mathrm{H}}^{\perp} = \frac{1}{2}\mathcal{T}^{MN}\delta_{\mathcal{B}}g_{MN} + \mathcal{J}^M\delta_{\mathcal{B}}A_M + \mathcal{K}\delta_{\mathcal{B}}\varphi + \mathcal{D} + \mathcal{O}(\partial^3). \tag{3.23}$$

We will now attempt to find all the solutions to this equation, hence recovering the null superfluid constitutive relations up to the first order in derivatives.

1. Parity-even

We can find the most general parity-even solution of Eq. (3.23) in two steps (note that $\mathrm{N}_{\mathrm{H}}^{\perp}$ is parity odd): (1) first, we write down the most general allowed parity-even $\mathcal{N}^M$ and find a set of constitutive relations pertaining to that, and (2) we find the most general parity-even constitutive relations which satisfy Eq. (3.23) with $\mathcal{N}^M = 0$.

(1) One can check that the most general form of $\mathcal{N}^M$ (whose divergence only contains product of

[7]Null and Newton-Cartan geometries behave more naturally in presence of a minimal temporal torsion $H_{MN} = 2\partial_{[M}V_{N]}$ (see [35]). In presence of H_{MN}, the data $S_2 = \zeta^M(\frac{1}{T}\partial_M T + u^N H_{NM})$ vanishes at equilibrium while $S_{e,1} = \frac{1}{T}\zeta^M\partial_M T$ survives. However, when $H_{MN} = 0$, $S_2 = S_{e,1}$.

TABLE IV. Independent null superfluid data at the first order in derivatives. Note that we have not, neither do we need to, enlist all the independent data that survives in equilibrium; the ones listed here are the only ones we use in the null superfluid constitutive relations.

Newton-Cartan data		Noncovariant data	
Vanishing at equilibrium—On-shell independent			
S_1	$\tilde{p}^{\mu}{}_{\nu}\nabla_{\mu}u^{\nu}$	S_1	$\tilde{p}^{ij}\partial_i u_j$
$S_2,\ S_{e,1}$	$\frac{1}{T}\zeta^{\mu}\partial_{\mu}T$	$S_2,\ S_{e,1}$	$\frac{1}{T}\zeta^{i}\partial_i T$
S_3	$\zeta^{\mu}\zeta_{\nu}\nabla_{\mu}u^{\nu}$	S_3	$\zeta^i\zeta_j\partial_i u^j$
S_4	$\zeta^{\mu}(T\partial_{\mu}\nu + u^{\nu}F_{\nu\mu})$	S_4	$\zeta^i(T\partial_i\nu - e_i + u^j\beta_{ji})$
S_5	$-\frac{1}{2}\zeta^{\mu}\zeta_{\mu} - \mu_s + \mu_n - \mu$	S_5	$-\frac{1}{2}\zeta^k\zeta_k - \mu_s + \mu_n - \mu$
$V_1^{\mu}, V_{e,1}^{\mu}$	$\frac{1}{T}\tilde{p}^{\mu\nu}\partial_{\nu}T$	$V_1^i, V_{e,1}^i$	$\frac{1}{T}\tilde{p}^{ij}\partial_j T$
V_2^{μ}	$2\tilde{p}^{\mu\nu}\zeta^{\sigma}p_{\rho(\sigma}\nabla_{\nu)}u^{\rho}$	V_2^i	$\tilde{p}^{ij}\zeta^k\partial_{(j}u_{k)}$
V_3^{μ}	$\tilde{p}^{\mu\nu}(T\partial_{\nu}\nu + u^{\rho}F_{\rho\nu})$	V_3^i	$\tilde{p}^{ij}(T\partial_j\nu - e_j + u^k\beta_{kj})$
$\sigma^{\mu\nu}$	$\tilde{p}^{\mu\rho}\tilde{p}^{\nu\sigma}\left(p_{\tau(\rho}\nabla_{\sigma)}u^{\tau} - \frac{\tilde{p}_{\rho\sigma}}{d-1}S_1\right)$	σ^{ij}	$\tilde{p}^{ik}\tilde{p}^{jl}(\partial_{(k}u_{l)} + \frac{\tilde{p}_{kl}}{d-1}S_1)$
$\tilde{V}_1^{\mu}$	$-\varepsilon^{\mu\nu\rho\sigma}n_{\nu}\zeta_{\rho}V_{1,\sigma}$	$\tilde{V}_1^i$	$\varepsilon^{ijk}\zeta_j V_{1,k}$
$\tilde{V}_2^{\mu}$	$-\varepsilon^{\mu\nu\rho\sigma}n_{\nu}\zeta_{\rho}V_{2,\sigma}$	$\tilde{V}_2^i$	$\varepsilon^{ijk}\zeta_j V_{2,k}$
$\tilde{V}_3^{\mu}$	$-\varepsilon^{\mu\nu\rho\sigma}n_{\nu}\zeta_{\rho}V_{3,\sigma}$	$\tilde{V}_3^i$	$\varepsilon^{ijk}\zeta_j V_{3,k}$
$\tilde{\sigma}^{\mu\nu}$	$-\varepsilon^{(\mu\vert\rho\sigma\tau}n_{\rho}\zeta_{\sigma}\sigma_{\tau}^{\nu)}$	$\tilde{\sigma}^{ij}$	$\varepsilon^{(i\vert kl}\zeta_k\sigma_l^{j)}$
Vanishing at equilibrium—On-shell dependent			
S_6	$\frac{1}{T}u^{\mu}\partial_{\mu}T$	S_6	$\frac{1}{T}(\partial_t T + u^i\partial_i T)$
S_7	$Tu^{\mu}\partial_{\mu}\nu$	S_7	$T(\partial_t\nu + u^i\partial_i\nu)$
S_8	$Tu^{\mu}\partial_{\mu}\nu_n$	S_8	$T(\partial_t\nu_n + u^i\partial_i\nu_n)$
S_9	$\zeta^{\mu}\left(T\partial_{\mu}\nu_n + u^{\nu}p_{\rho\mu}\nabla_{\nu}u^{\rho}\right)$	S_9	$\zeta^i(T\partial_i\nu_n + \partial_t u_i + u^j\partial_j u_i)$
V_4^{μ}	$\tilde{P}^{\mu\nu}\left(T\partial_{\nu}\nu_n + u^{\sigma}p_{\rho\nu}\nabla_{\sigma}u^{\rho}\right)$	V_4^i	$\tilde{p}^{ij}(T\partial_j\nu_n + \partial_t u_j + u^k\partial_k u_j)$
$\tilde{V}_4^{\mu}$	$-\varepsilon^{\mu\nu\rho\sigma}n_{\nu}\zeta_{\rho}V_{4,\sigma}$	$\tilde{V}_4^i$	$\varepsilon^{ijk}\zeta_j V_{4,k}$
Surviving at equilibrium			
$S_{e,2}$	$T\zeta^{\mu}\partial_{\mu}\nu$	$S_{e,2}$	$T\zeta^i\partial_i\nu$
$S_{e,3}$	$T\zeta^{\mu}\partial_{\mu}\nu_n$	$S_{e,3}$	$T\zeta^i\partial_i\nu_n$
$V_{e,2}^{\mu}$	$T\tilde{p}^{\mu\nu}\partial_{\nu}\nu$	$V_{e,2}^i$	$T\tilde{p}^{ij}\partial_j\nu$
$V_{e,3}^{\mu}$	$T\tilde{P}^{\mu\nu}\partial_{\nu}\nu_n$	$V_{e,3}^i$	$T\tilde{p}^{ij}\partial_j\nu_n$
$\tilde{S}_{e,1}$	$T\varepsilon^{\mu\nu\rho\sigma}n_{\mu}\zeta_{\nu}\partial_{\rho}B_{\sigma}$	$\tilde{S}_{e,1}$	$T\varepsilon^{ijk}\zeta_i\partial_j u_k$
$\tilde{S}_{e,2}$	$\frac{T}{2}\varepsilon^{\mu\nu\rho\sigma}n_{\mu}\zeta_{\nu}F_{\rho\sigma}$	$\tilde{S}_{e,2}$	$\frac{T}{2}\varepsilon^{ijk}\zeta_i\beta_{jk}$
$\tilde{V}_{e,1}^{\mu}$	$-T\tilde{p}_{\tau}^{\mu}\varepsilon^{\tau\nu\rho\sigma}n_{\nu}\partial_{\rho}B_{\sigma}$	$\tilde{V}_{e,1}^i$	$T\tilde{p}_l^i\varepsilon^{ljk}\partial_j u_k$
$\tilde{V}_{e,2}^{\mu}$	$-\frac{T}{2}\tilde{p}_{\tau}^{\mu}\varepsilon^{\tau\nu\rho\sigma}n_{\nu}F_{\rho\sigma}$	$\tilde{V}_{e,2}^i$	$\frac{T}{2}\tilde{p}_l^i\varepsilon^{ljk}\beta_{jk}$
$\tilde{V}_{e,3}^{\mu}$	$T\tilde{p}_{\tau}^{\mu}\varepsilon^{\tau\nu\rho\sigma}\zeta_{\nu}\partial_{\rho}B_{\sigma} + (\mu_s + \frac{1}{2}\zeta^{\mu}\zeta_{\mu})\tilde{V}_{e,1}^{\mu}$	$\tilde{V}_{e,3}^i$	$-T(u^i\varepsilon^{jkl}\zeta_j\partial_k u_l - \varepsilon^{ijk}\zeta_j\partial_t u_k) + (\mu_s + \frac{1}{2}\zeta^k\zeta_k)\tilde{V}_{e,1}^i$
$\tilde{V}_{e,4}^{\mu}$	$\frac{T}{2}\tilde{p}_{\tau}^{\mu}\varepsilon^{\tau\nu\rho\sigma}\zeta_{\nu}F_{\rho\sigma} + (\mu_s + \frac{1}{2}\zeta^{\mu}\zeta_{\mu})\tilde{V}_{e,2}^{\mu}$	$\tilde{V}_{e,4}^i$	$-T(u^i\frac{1}{2}\varepsilon^{jkl}\zeta_j\beta_{kl} + \varepsilon^{ijk}\zeta_j e_k) + (\mu_s + \frac{1}{2}\zeta^k\zeta_k)\tilde{V}_{e,2}^i$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

derivatives and has at least one $\delta_{\mathcal{B}}$ per term) can be written as (see Appendix B for more details),

$$\mathcal{N}^M = 2f_1 u^{[M}\zeta^{N]}\frac{1}{T^2}\partial_N T + 2f_2 u^{[M}\zeta^{N]}\partial_N\nu + 2f_3 u^{[M}\zeta^{N]}\partial_N\nu_n + 2f_4 u^{[M}\zeta^{N]}\partial_N R_s + \nabla_N(f_5 u^{[M}\zeta^{N]}), \tag{3.24}$$

where f's are functions of T, $\nu = \mu/T$, $\nu_n = \mu_n/T$ and $\hat{\mu}_s = -\frac{1}{2}\zeta^M\zeta_M$ with $\zeta^M = P^{MN}\xi_N = \xi^M - u^M + (u^N\xi_N)V^M$ ($P^{MN} = g^{MN} + 2u^{(M}V^{N)}$ is the projection operator away from the null fluid velocity). Note that

$$\hat{\mu}_s = -\frac{1}{2}\zeta^M\zeta_M = -\frac{1}{2}\xi^M\xi_M + \xi^M u_M = \mu_s + \xi^M u_M = \mu_s - \mu_n + \mu + T\delta_{\mathcal{B}}\varphi. \tag{3.25}$$

Out of the five terms in Eq. (3.24), the last one has trivially zero divergence and hence can be ignored. The forth term on the other hand can be removed by elimination of $\nabla_M(R_s\xi^M)$ using the φ equation of motion. Computing the divergence of the remaining terms in $\mathcal{N}^M$ and comparing them to Eq. (3.23), we can directly read out the corresponding null superfluid constitutive relations (the symbol '$\ni$' represents that they are not yet the complete solutions of Eq. (3.23); we still have to add the terms with $\mathcal{N}^M = 0$),

$$
\begin{aligned}
\mathcal{T}^{MN} \ni u^M u^N \left(\sum_{i=1}^{3} \alpha_{R_n,i} S_{e,i} - \frac{1}{T} \nabla_R (T f_3 \zeta^R) \right) + 2V^{(M} u^{N)} \left(\sum_{i=1}^{3} \alpha_{E,i} S_{e,i} - \frac{1}{T} \nabla_R (T f_1 \zeta^R) \right) \\
+ (\zeta^M \zeta^N + 2\zeta^{(M} u^{N)} - 2\zeta^{(M} V^{N)} (u^R \xi_R)) \sum_{i=1}^{3} \alpha_{R_s,i} S_{e,i} - 2\xi^{(M} \sum_{i=1}^{3} f_i V_{e,i}^{N)} + \tilde{P}^{MN} \sum_{i=1}^{3} f_i S_{e,i} + 2\zeta^{(M} V^{N)} \sum_{i=1}^{3} f_i S_{5+i}, \\
\mathcal{J}^M \ni u^M \left(\sum_{i=1}^{3} \alpha_{Q,i} S_{e,i} - \frac{1}{T} \nabla_R (T f_2 \zeta^R) \right) - \zeta^M \sum_{i=1}^{3} \alpha_{R_s,i} S_{e,i} + \sum_{i=1}^{3} f_i V_{e,i}^M, \\
\mathcal{K} \ni \nabla_M \left(\zeta^M \sum_{i=1}^{3} \alpha_{R_s,i} S_{e,i} - \sum_{i=1}^{3} f_i V_{e,i}^M \right),
\end{aligned}
\tag{3.26}
$$

where $\tilde{P}^{MN} = g^{\mu\nu} + 2u^{(M} V^{N)} - \frac{1}{\zeta^R \zeta_R} \zeta^M \zeta^N$, and we have defined,

$$
\mathrm{d}f_i = \frac{\alpha_{E,i}}{T} \mathrm{d}T + T\alpha_{R_n,i} \mathrm{d}\nu_n + T\alpha_{Q,i} \mathrm{d}\nu + \left(\alpha_{R_s,i} - \frac{f_i}{2\hat{\mu}_s} \right) \mathrm{d}\hat{\mu}_s. \tag{3.27}
$$

The actual computation is not neat and we have presented the details in Appendix C for the aid of the readers interested in reproducing our results. Note that these constitutive relations are presented in terms of "data" which are natural for this sector; readers can modify these to their favorite basis and get results which might look considerably messier. Moreover, these results are written in a particular "hydrodynamic frame" chosen by aligning u^M, T, μ, μ_n along β^M, Λ_β, which again can be modified according to reader's preference.

(2) Let us now look at the parity-even solutions to Eq. (3.23) with $\mathcal{N}^M = 0$,

$$
0 = \frac{1}{2} \mathcal{T}^{MN} \delta_{\mathcal{B}} g_{MN} + \mathcal{J}^M \delta_{\mathcal{B}} A_M + \mathcal{K} \delta_{\mathcal{B}} \varphi + \mathcal{D}. \tag{3.28}
$$

Every term in $\mathcal{T}^{MN}, \mathcal{J}^M, \mathcal{K}$ must either cancel or contribute to Δ which has to be a quadratic form. It follows that the terms in $\mathcal{T}^{MN}, \mathcal{J}^M, \mathcal{K}$ must be proportional to $\delta_{\mathcal{B}} g_{MN}, \delta_{\mathcal{B}} A_M, \delta_{\mathcal{B}} \varphi$. Recall, however, that we have chosen to eliminate $u^M \delta_{\mathcal{B}} g_{MN}$, $u^M \delta_{\mathcal{B}} A_M$ using the equations of motion. For Δ to be a quadratic form, it therefore implies that $\mathcal{T}^{MN}, \mathcal{J}^M$ cannot have a term like $\#^{(M} u^{N)}$, $\# u^M$ respectively for some vector $\#^M$ and scalar #. With this input let us write down the most generic allowed form of the currents in terms of 34 new transport coefficients $[\beta_{ij}]_{5\times5}$ (with $\beta_{55} = \alpha/T$), $[\kappa_{ij}]_{3\times3}$ and η,

$$
\begin{aligned}
\mathcal{T}^{MN} \ni & -T[\{\beta_{11} \tilde{P}^{RS} + 2\beta_{12} \zeta^{(R} V^{S)} + \beta_{13} \zeta^R \zeta^S\} \tilde{P}^{MN} + \{\beta_{21} \tilde{P}^{RS} + 2\beta_{22} \zeta^{(R} V^{S)} + \beta_{23} \zeta^R \zeta^S\} 2\zeta^{(M} V^{N)} \\
& + \{\beta_{31} \tilde{P}^{RS} + 2\beta_{32} \zeta^{(R} V^{S)} + \beta_{33} \zeta^R \zeta^S\} \zeta^M \zeta^N \\
& + 4\{\kappa_{11} V^{(R} + \kappa_{12} \zeta^{(R}\} \tilde{P}^{S)(M} V^{N)} + 4\{\kappa_{21} V^{(R} + \kappa_{22} \zeta^{(R}\} \tilde{P}^{S)(M} \zeta^{N)} + \eta \tilde{P}^{M\langle R} P^{S\rangle N}] \frac{1}{2} \delta_{\mathcal{B}} g_{RS} \\
& - T[\beta_{14} \zeta^R \tilde{P}^{MN} + 2\beta_{24} \zeta^R \zeta^{(M} V^{N)} + \beta_{34} \zeta^R \zeta^M \zeta^N + 2\kappa_{13} \tilde{P}^{R(M} V^{N)} + 2\kappa_{23} \tilde{P}^{R(M} \zeta^{N)}] \delta_{\mathcal{B}} A_R, \\
& - T[\beta_{15} \tilde{P}^{MN} + 2\beta_{25} \zeta^{(M} V^{N)} + \beta_{35} \zeta^M \zeta^N] \delta_{\mathcal{B}} \varphi \\
= & -\tilde{P}^{MN} \sum_{i=1}^{5} \beta_{1i} S_i - 2\zeta^{(M} V^{N)} \sum_{i=1}^{5} \beta_{2i} S_i - \zeta^M \zeta^N \sum_{i=1}^{5} \beta_{3i} S_i - 2V^{(M} \sum_{i=1}^{3} \kappa_{1i} V_i^{N)} \\
& - 2\zeta^{(M} \sum_{i=1}^{3} \kappa_{2i} V_i^{N)} - \eta \sigma^{MN},
\end{aligned}
\tag{3.29}
$$

$$\mathcal{J}^M \ni -T[\{\beta_{41}\tilde{P}^{RS} + 2\beta_{42}\zeta^{(R}V^{S)} + \beta_{43}\zeta^R\zeta^S\}\zeta^M + 2\{\kappa_{31}V^{(R} + \kappa_{32}\zeta^{(R}\}\tilde{P}^{S)M}]\frac{1}{2}\delta_{\mathcal{B}}g_{RS}$$
$$- T[\beta_{44}\zeta^M\zeta^N + \kappa_{33}P^{MN}]\delta_{\mathcal{B}}A_R - T[\beta_{45}\zeta^M]\delta_{\mathcal{B}}\varphi,$$
$$= -\zeta^M\sum_{i=1}^{5}\beta_{4i}S_i - \sum_{i=1}^{3}\kappa_{3i}V_i^M, \tag{3.30}$$

$$\mathcal{K} \ni -T[\beta_{51}\tilde{P}^{RS} + 2\beta_{52}\zeta^{(R}V^{S)} + \beta_{53}\zeta^R\zeta^S]\delta_{\mathcal{B}}g_{RS} - T[\beta_{54}\zeta^M]\delta_{\mathcal{B}}A_M = -\sum_{i=1}^{4}\beta_{5i}S_i. \tag{3.31}$$

Note that we did not include a term proportional to $\delta_{\mathcal{B}}\varphi$ in $\mathcal{K}$, because such a term is already present in $K = -\alpha\delta_{\mathcal{B}}\varphi + \nabla_M(R_s\xi^M) + \mathcal{K} + \mathcal{O}(\partial^2)$. Plugging these back into Eq. (3.28) and defining $\beta_{55} = \alpha/T$ we can read out the parity-even quadratic form $\Delta|_{\text{even}} = \alpha(\delta_{\mathcal{B}}\varphi)^2 + \mathcal{D}|_{\text{even}}$,

$$T\Delta|_{\text{even}} = \sum_{i,j=1}^{5} S_i\beta_{ij}S_j + \sum_{i,j=1}^{3} V_i^M\kappa_{ij}V_{j,M} + \eta\sigma^{MN}\sigma_{MN},$$
$$= \sum_{i,j=1}^{5} S_i\beta_{(ij)}S_j + \sum_{i,j=1}^{3} V_i^M\kappa_{(ij)}V_{j,M} + \eta\sigma^{MN}\sigma_{MN}. \tag{3.32}$$

In the second step we have realized that only the symmetric parts of the matrices β_{ij} and κ_{ij} will survive in this expression, and will contribute towards dissipation. Thus only 22 out of 35 transport coefficients (including α) are dissipative; the remaining 13 are nondissipative.

2. Parity-odd (five dimensions)

We can find the most general parity-odd solution of Eq. (3.23) in three steps: (1) first we consider a particular set of solutions which takes care of the anomaly $\mathrm{N}_{\mathrm{H}}^{\perp}$ and proceed towards the nonanomalous constitutive relations, (2) then we write down the most general allowed parity-odd $\mathcal{N}^M$ and find a set of constitutive relations pertaining to that, and (2) finally find the most general parity-odd constitutive relations with zero $\mathcal{N}^M$.

(1) In four dimensions at first order in the derivatives $\mathrm{T}_{\mathrm{H}}^{M\perp} = 0$ and $\mathrm{J}_{\mathrm{H}}^{\perp} = -\frac{3}{4}C^{(4)}\epsilon^{MNRST}u_M F_{NR}F_{SR}$ [17,22], which implies,

$$\mathrm{N}_{\mathrm{H}}^{\perp} = -\frac{3}{4}C^{(4)}\frac{\mu}{T}\epsilon^{MNRST}u_M F_{NR}F_{SR}. \tag{3.33}$$

A particular solution pertaining to Eq. (3.23) with this $\mathrm{N}_{\mathrm{H}}^{\perp}$ is given as (see [17]),

$$\mathcal{T}^{MN} \ni 6C^{(4)}\mu^2 V^{(M}M^{N)}, \qquad \mathcal{J}^M \ni 6C^{(4)}\mu M^M,$$
$$\mathcal{K} \ni 0, \qquad \mathcal{N}^M \ni 3C^{(4)}\frac{\mu^2}{T}M^M. \tag{3.34}$$

Here we have defined the magnetic field and fluid vorticity as

$$M^M = \frac{1}{2}\epsilon^{MNRST}V_N u_R F_{ST},$$
$$\omega^M = \epsilon^{MNRS\sigma}V_N u_R\partial_S u_T. \tag{3.35}$$

(2) One can check that the most general form of $\mathcal{N}^M$ (whose divergence only contains the product of derivatives and has at least one $\delta_{\mathcal{B}}$ per term) can be written as (see Appendix B for more details),

$$\mathcal{N}^M = g_1(\beta^M\tilde{S}_{e,1} + \tilde{V}_4^M) + g_2(\beta^M\tilde{S}_{e,2} + \tilde{V}_3^M)$$
$$+ g_3\tilde{V}_1^M + C_1 T\omega^M, \tag{3.36}$$

where g's are functions of T, ν, $\hat{\mu}_s$, and C_1 is a constant[8] From here we can directly read out the corresponding constitutive relations,

[8] It might be noted that since $\nabla_M\omega^M = 0$, C_1 a priory can be an arbitrary function rather than a constant. However, if we do the same computation in presence of torsion and later turn it off, which allows for $\partial_{[M}V_{N]} \neq 0$, we will be forced to set C_1 to be a constant (see Appendix (A) of [17]). Another way to see that C_1 should be a constant is using the equilibrium partition function discussed in Appendix B.

$$\begin{aligned}
\mathcal{T}^{MN} \ni\ & u^M u^N \sum_{i=1}^{2} \tilde{\alpha}_{R_n,i} \tilde{S}_{e,i} + 2V^{(M} u^{N)} \sum_{i=1}^{2} \tilde{\alpha}_{E,i} \tilde{S}_{e,i} \\
& + (\zeta^M \zeta^N + 2\zeta^{(M} u^{N)} - 2\zeta^{(M} V^{N)} (u^R \xi_R)) \sum_{i=1}^{2} \tilde{\alpha}_{R_s,i} \tilde{S}_{e,i} - \zeta^M \zeta^N \sum_{i=1}^{2} \frac{g_i}{2\hat{\mu}_s} \tilde{S}_{e,i} \\
& - 2V^{(M} \sum_{i=1}^{2} g_i \tilde{V}_{e,i+2}^{N)} - 2u^{(M} \sum_{i=1}^{2} g_i \tilde{V}_{e,i}^{N)} + 2C_1 T^2 V^{(M} \omega^{N)} \\
& + 2u^{(M} P_P^{N)} \epsilon^{PKRST} \nabla_K (T g_1 V_R u_S \xi_T) + 2V^{(M} P_P^{N)} \nabla_K (g_3 T \epsilon^{PKRST} V_R u_S \zeta_T), \\
\mathcal{J}^M \ni\ & u^M \sum_{i=1}^{2} \tilde{\alpha}_{Q,i} \tilde{S}_{e,i} - \zeta^M \sum_{i=1}^{2} \tilde{\alpha}_{R_s,i} \tilde{S}_{e,i} + \sum_{i=1}^{2} g_i \tilde{V}_{e,i}^M + P_K^M \epsilon^{KNRST} \nabla_N (T g_2 V_R u_S \xi_T), \\
\mathcal{K} \ni\ & \nabla_M \left(\zeta^M \sum_{i=1}^{2} \tilde{\alpha}_{R_s,i} \tilde{S}_{e,i} - \sum_{i=1}^{2} g_i \tilde{V}_{e,i}^M \right),
\end{aligned} \tag{3.37}$$

where we have defined,

$$\mathrm{d}g_i = \frac{1}{T} \tilde{\alpha}_{E,i} \mathrm{d}T + T \tilde{\alpha}_{Q,i} \mathrm{d}\nu + T \tilde{\alpha}_{R_n,i} \mathrm{d}\nu_n + \left(\tilde{\alpha}_{R_s,i} - \frac{g_i}{2\hat{\mu}_s} \right) \mathrm{d}\hat{\mu}_s. \tag{3.38}$$

The actual computation is not neat and we have presented the details in Appendix C for interested readers.

(3) We should finally consider the parity-odd constitutive relations that satisfy Eq. (3.23) with zero lhs. Following our discussion in the parity-even sector, the allowed form of the constitutive relations can be written down in terms of 10 coefficients $[\tilde{\kappa}_{ij}]_{3\times 3}$ and $\tilde{\eta}$,

$$\begin{aligned}
\mathcal{T}^{MN} \ni\ & -T V_T u_K \zeta_L [4V^{(M} \epsilon^{N)TKL(R} \{\tilde{\kappa}_{11} V^{S)} + \tilde{\kappa}_{12} \zeta^{S)}\} + 4\zeta^{(M} \epsilon^{N)TKL(R} \{\tilde{\kappa}_{21} V^{S)} + \tilde{\kappa}_{22} \zeta^{S)}\} \\
& + \tilde{\eta} \tilde{P}^{P(M} \epsilon^{N)TKL(R} \tilde{P}_P^{S)}] \frac{1}{2} \delta_{\mathcal{B}} g_{RS} - T V_T u_K \zeta_L [2\tilde{\kappa}_{13} V^{(M} \epsilon^{N)TKLR} + 2\tilde{\kappa}_{23} \zeta^{(M} \epsilon^{N)TKLR}] \delta_{\mathcal{B}} A_R, \\
=\ & -2V^{(M} \sum_{i=1}^{3} \tilde{\kappa}_{1i} \tilde{V}_i^{N)} - 2\zeta^{(M} \sum_{i=1}^{3} \tilde{\kappa}_{2i} \tilde{V}_i^{N)} - \tilde{\eta} \tilde{\sigma}^{MN}, \\
\mathcal{J}^\mu \ni\ & -T V_T u_K \zeta_L [2\epsilon^{MTKL(R} \{\tilde{\kappa}_{31} V^{S)} + \tilde{\kappa}_{32} \zeta^{S)}\}] \frac{1}{2} \delta_{\mathcal{B}} g_{RS} - T V_T u_K \zeta_L [\tilde{\kappa}_{33} \epsilon^{MTKLR}] \delta_{\mathcal{B}} A_R, \\
=\ & -\sum_{i=1}^{3} \tilde{\kappa}_{3i} \tilde{V}_i^M, \\
\mathcal{K} \ni\ & 0.
\end{aligned} \tag{3.39}$$

One can check that these constitutive relations trivially satisfy Eq. (3.23) with zero lhs and the quadratic form $\Delta|_{\mathrm{odd}} = \mathcal{D}|_{\mathrm{odd}}$ is given as

$$\begin{aligned}
T\Delta|_{\mathrm{odd}} \ni\ & -\epsilon^{MNRST} V_R u_S \zeta_T \left[\sum_{i,j=1}^{3} V_{i,M} \tilde{\kappa}_{ij} V_{j,N} + \tilde{\eta} \sigma_{MP} \sigma_N^P \right], \\
=\ & -\epsilon^{MNRST} V_R u_S \zeta_T \sum_{i,j=1}^{3} V_{i,M} \tilde{\kappa}_{ij}^{(a)} V_{j,N}.
\end{aligned} \tag{3.40}$$

It follows that out of the 10 transport coefficients, only 3 contribute to dissipation and the other 7 are nondissipative.

3. Positivity constraints

The dissipative transport coefficients are required to satisfy a set of inequalities to agree with $\Delta = \alpha(\delta_{\mathcal{B}}\varphi)^2 + \mathcal{D}|_{\text{even}} + \mathcal{D}|_{\text{odd}} \geq 0$,

$$T\Delta = \sum_{i,j=1}^{5} S_i \beta_{(ij)} S_j + \left(\sum_{i,j=1}^{3} V_i^M \kappa_{(ij)} V_{j,M} + \sum_{i,j=1}^{3} V_i^M \tilde{\kappa}_{[ij]} \tilde{V}_{j,M}\right) + \eta \sigma^{MN} \sigma_{MN}. \tag{3.41}$$

We want this expression to be a quadratic form, which it nearly is except the parity-odd terms in the brackets. However this term can be made into a quadratic form by noting that the square of a parity-odd term is parity-even, due to the identity,

$$(\epsilon^{MNRST} V_R u_S \zeta_T)(\epsilon_{MKLOP} V^L u^O \zeta^P) = \tilde{P}_K^N \zeta^M \zeta_M = -2\hat{\mu}_s \tilde{P}_K^N. \tag{3.42}$$

We define

$$\begin{pmatrix} V_1'^M \\ V_2'^M \\ V_3'^M \end{pmatrix} = \begin{pmatrix} V_1^M \\ V_2^M \\ V_3^M \end{pmatrix} + \begin{pmatrix} 0 & a_{12} & a_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \tilde{V}_1'^M \\ \tilde{V}_2'^M \\ \tilde{V}_3'^M \end{pmatrix},$$
$$\kappa'_{ij} = \kappa_{ij} + k_{ij}, \quad k_{[ij]} = 0, \tag{3.43}$$

such that

$$\sum_{i,j=1}^{3} V_i'^M \kappa'_{(ij)} V'_{j,M} = \sum_{i,j=1}^{3} V_i^M \kappa_{(ij)} V_{j,M} + \sum_{i,j=1}^{3} V_i^M \tilde{\kappa}_{[ij]} \tilde{V}_{j,M}. \tag{3.44}$$

Using the identity Eq. (3.42), the above equation can be easily solved to give

$$[a_{ij}] = \begin{pmatrix} 0 & \frac{\tilde{\kappa}_{[12]}}{\kappa_{11}} & \frac{\kappa_{11}(\kappa_{22}\tilde{\kappa}_{[13]} - \kappa_{(12)}\tilde{\kappa}_{[23]}) - \tilde{\kappa}_{[12]}(\kappa_{(12)}\kappa_{(13)} + \zeta^M \zeta_M \tilde{\kappa}_{[12]}\tilde{\kappa}_{[13]})}{\kappa_{11}(\kappa_{11}\kappa_{22} - \kappa_{(12)}^2 - \zeta^M \zeta_M \tilde{\kappa}_{[12]}^2)} \\ 0 & 0 & \frac{\kappa_{11}\tilde{\kappa}_{[23]} - \kappa_{(12)}\tilde{\kappa}_{[13]} + \kappa_{(13)}\tilde{\kappa}_{[12]}}{\kappa_{11}\kappa_{22} - \kappa_{(12)}^2 - \zeta^M \zeta_M \tilde{\kappa}_{[12]}^2} \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.45}$$

$$[k_{ij}] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\zeta^M \zeta_M \frac{(\tilde{\kappa}_{[12]})^2}{\kappa_{11}} & -\zeta^M \zeta_M \frac{\tilde{\kappa}_{[12]}\tilde{\kappa}_{[13]}}{\kappa_{11}} \\ 0 & -\zeta^M \zeta_M \frac{\tilde{\kappa}_{[12]}\tilde{\kappa}_{[13]}}{\kappa_{11}} & -\zeta^M \zeta_M \left(\frac{(\tilde{\kappa}_{[13]})^2}{\kappa_{11}} + \frac{(\kappa_{11}\tilde{\kappa}_{[23]} - \kappa_{(12)}\tilde{\kappa}_{[13]} + \kappa_{(13)}\tilde{\kappa}_{[12]})^2}{\kappa_{11}(\kappa_{11}\kappa_{22} - \kappa_{(12)}^2 - \zeta^M \zeta_M \tilde{\kappa}_{[12]}^2)}\right) \end{pmatrix}. \tag{3.46}$$

Consequently Δ will take the form

$$T\Delta = \sum_{i,j=1}^{5} S_i \beta_{(ij)} S_j + \sum_{i,j=1}^{3} V_i'^M \kappa'_{(ij)} V'_{j,M} + \eta \sigma^{MN} \sigma_{MN}. \tag{3.47}$$

Given $T \geq 0$, the condition $\Delta \geq 0$ implies that $\eta \geq 0$ and the matrices $[\beta_{(ij)}]_{5\times 5}$, $[\kappa'_{(ij)}]_{3\times 3}$ have all non-negative eigenvalues. This gives 9 inequalities among 25 dissipative transport coefficients, and 16 are completely arbitrary.

E. Summary

We have completed the analysis of a null superfluid up to the first order in derivatives. Here we summarize the results. We found that the entire null superfluid transport up to the first order in derivatives is characterized by an ideal order pressure P, 51 first-order transport coefficients which are functions of T, μ/T, μ_n/T, $\hat{\mu}_s$, and two constants C_1, $C^{(4)}$. P, C_1 and $C^{(4)}$ along with 6 transport coefficients,

$$\text{parity even (3):} \quad f_1, \quad f_2, \quad f_3,$$
$$\text{parity odd (3):} \quad g_1, \quad g_2, \quad g_3, \tag{3.48}$$

totally determine the hydrostatic transport (part of the constitutive relations that survive at equilibrium). Nonhydrostatic nondissipative transport (part that does not survive at equilibrium but doesn't contribute to $\Delta \geq 0$ either) is given by 20 transport coefficients,

$$\text{parity even (13):} \quad [\beta_{[ij]}]_{5\times 5} \quad \text{(antisymmetric)},$$
$$[\kappa_{[ij]}]_{3\times 3} \quad \text{(antisymmetric)},$$
$$\text{parity odd (7):} \quad [\tilde{\kappa}_{(ij)}]_{3\times 3} \quad \text{(symmetric)}, \qquad \tilde{\eta}. \tag{3.49}$$

Finally the entire dissipative transport is given by 25 transport coefficients,

$$\begin{aligned} &\text{parity even}\,(22)\colon\ [\beta_{(ij)}]_{5\times 5}\quad(\text{symmetric}),\qquad [\kappa_{(ij)}]_{3\times 3}\quad(\text{symmetric}),\qquad \eta,\\ &\text{parity odd}\,(3)\colon\ [\tilde{\kappa}_{[ij]}]_{3\times 3}\quad(\text{antisymmetric}). \end{aligned} \tag{3.50}$$

These dissipative transport coefficients follow a set of inequalities [κ'_{ij} is defined in Eq. (3.43)],

$$[\beta_{(ij)}]_{5\times 5},\qquad [\kappa'_{(ij)}]_{3\times 3},\qquad \eta\geq 0, \tag{3.51}$$

where a "non-negative matrix" implies all its eigenvalues are non-negative.

Out of these for an ordinary null fluid, shear viscosity η, bulk viscosity $\zeta=\beta_{11}$, conductivities κ_{11}, κ_{13}, κ_{31}, κ_{33} and the constants C_1, $C^{(4)}$ are present. In addition g_1, g_2 and g_3 are forced to be constants, while all the remaining transport coefficients zero.

Using P, f_i, g_i we define some new functions,

$$\begin{aligned} &\mathrm{d}P=S\mathrm{d}T+Q\mathrm{d}\mu+R_n\mathrm{d}\mu_n+R_s\mathrm{d}\mu_s,\qquad E+P=ST+Q\mu+R_n\mu_n,\\ &\mathrm{d}f_i=\frac{\alpha_{E,i}}{T}\mathrm{d}T+T\alpha_{R_n,i}\mathrm{d}\nu_n+T\alpha_{Q,i}\mathrm{d}\nu+\left(\alpha_{R_s,i}-\frac{f_i}{2\hat{\mu}_s}\right)\mathrm{d}\hat{\mu}_s,\qquad \alpha_{E,i}+f_i=\alpha_{S,i}T+\alpha_{Q,i}\mu+\alpha_{R_n,i}\mu_n,\\ &\mathrm{d}g_i=\frac{\tilde{\alpha}_{E,i}}{T}\mathrm{d}T+T\tilde{\alpha}_{R_n,i}\mathrm{d}\nu_n+T\tilde{\alpha}_{Q,i}\mathrm{d}\nu+\left(\tilde{\alpha}_{R_s,i}-\frac{g_i}{2\hat{\mu}_s}\right)\mathrm{d}\hat{\mu}_s,\qquad \tilde{\alpha}_{E,i}+g_i=\tilde{\alpha}_{S,i}T+\tilde{\alpha}_{Q,i}\mu+\tilde{\alpha}_{R_n,i}\mu_n. \end{aligned} \tag{3.52}$$

In terms of these transport coefficients, corrections to the Josephson equation ($K=0$) coming from the first-order null superfluid transport are given as (here $\beta_{55}=\alpha/T$),

$$\begin{aligned} u^M\xi_M+\mu_n-\mu=&\frac{1}{\beta_{55}}\nabla_M(R_s\xi^M)-\sum_{i=1}^{4}\frac{\beta_{5i}}{\beta_{55}}S_i\\ &+\frac{1}{\beta_{55}}\nabla_M\left(\zeta^M\sum_{i=1}^{3}\alpha_{R_s,i}S_{e,i}+\zeta^M\sum_{i=1}^{2}\tilde{\alpha}_{R_s,i}\tilde{S}_{e,i}-\sum_{i=1}^{3}f_iV^M_{e,i}-\sum_{i=1}^{2}g_i\tilde{V}^M_{e,i}\right)+\mathcal{O}(\partial^2), \end{aligned} \tag{3.53}$$

which can be seen as determining $u^M\xi_M$ in terms of the other null superfluid variables. Note that though this equation contains second-order terms, it is only correct up to the first order in derivatives, and will admit further corrections coming from higher-order null superfluid transport. The energy-momentum tensor, charge current and entropy current up to first order in derivatives are, however, given as

$$\begin{aligned} T^{MN}&=R_nu^Mu^N+2Eu^{(M}V^{N)}+PP^{MN}+R_s\xi^M\xi^N+\mathcal{T}^{MN}+\mathcal{O}(\partial^2),\\ J^M&=Qu^M-R_s\xi^M+\mathcal{J}^M+\mathcal{O}(\partial^2)\\ J^M_S&=Su^M+\mathcal{S}^M+\mathcal{O}(\partial^2), \end{aligned} \tag{3.54}$$

where the higher derivative corrections are,

$$\begin{aligned} \mathcal{T}^{MN}=&\,u^Mu^N\left[\sum_{i=1}^{3}\alpha_{R_n,i}S_{e,i}+\sum_{i=1}^{2}\tilde{\alpha}_{R_n,i}\tilde{S}_{e,i}-\frac{1}{T}\nabla_R(Tf_3\zeta^R)\right]\\ &+2V^{(M}u^{N)}\left[\sum_{i=1}^{3}\alpha_{E,i}S_{e,i}+\sum_{i=1}^{2}\tilde{\alpha}_{E,i}\tilde{S}_{e,i}-\frac{1}{T}\nabla_R(Tf_1\zeta^R)\right]\\ &+2\zeta^{(M}u^{N)}\left[\sum_{i=1}^{3}\alpha_{R_s,i}S_{e,i}+\sum_{i=1}^{2}\tilde{\alpha}_{R_s,i}\tilde{S}_{e,i}\right] \end{aligned}$$

$$
\begin{aligned}
&+2\zeta^{(M}V^{N)}\left[\sum_{i=1}^{3} f_i S_{5+i} - (u^R\xi_R)\left(\sum_{i=1}^{3}\alpha_{R_s,i}S_{e,i} + \sum_{i=1}^{2}\tilde{\alpha}_{R_s,i}\tilde{S}_{e,i}\right) - \sum_{i=1}^{5}\beta_{2i}S_i\right]\\
&+\zeta^M\zeta^N\left[\sum_{i=1}^{3}\alpha_{R_s,i}S_{e,i} + \sum_{i=1}^{2}\tilde{\alpha}_{R_s,i}\tilde{S}_{e,i} - \sum_{i=1}^{2}\frac{g_i}{2\hat{\mu}_s}\tilde{S}_{e,i} - \sum_{i=1}^{5}\beta_{3i}S_i\right]\\
&+2u^{(M}\left[-\sum_{i=1}^{3} f_i V_{e,i}^{N)} - \sum_{i=1}^{2} g_i\tilde{V}_{e,i}^{N)} + P_P^{N)}\epsilon^{PKRST}\nabla_K(Tg_1V_Ru_S\xi_T)\right]\\
&+2V^{(M}\left[(u^R\xi_R)\sum_{i=1}^{3} f_i V_{e,i}^{N)} - \sum_{i=1}^{2} g_i\tilde{V}_{e,i+2}^{N)} - \sum_{i=1}^{3}\kappa_{1i}V_i^{N)} - \sum_{i=1}^{3}\tilde{\kappa}_{1i}\tilde{V}_i^{N)} + 3C^{(4)}\mu^2M^{N)}\right.\\
&\left.+P_P^{N)}\epsilon^{PKRST}\nabla_K(Tg_3V_Ru_S\zeta_T) + C_1T^2\omega^{N)}\right] + \tilde{P}^{MN}\left[\sum_{i=1}^{3} f_i S_{e,i} - \sum_{i=1}^{5}\beta_{1i}S_i\right]\\
&-2\zeta^{(M}\left[\sum_{i=1}^{3} f_i V_{e,i}^{N)} + \sum_{i=1}^{3}\kappa_{2i}V_i^{N)} + \sum_{i=1}^{3}\tilde{\kappa}_{2i}\tilde{V}_i^{N)}\right] - \eta\sigma^{MN} - \tilde{\eta}\tilde{\sigma}^{MN},
\end{aligned}
\tag{3.55}
$$

$$
\begin{aligned}
\mathcal{J}^M = {}& u^M\left[\sum_{i=1}^{3}\alpha_{Q,i}S_{e,i} + \sum_{i=1}^{2}\tilde{\alpha}_{Q,i}\tilde{S}_{q,i} - \frac{1}{T}\nabla_R(Tf_2\zeta^R)\right] + P_K^M\epsilon^{KNRST}\nabla_N(Tg_2V_Ru_S\xi_T)\\
&-\zeta^M\left[\sum_{i=1}^{3}\alpha_{R_s,i}S_{e,i} + \sum_{i=1}^{2}\tilde{\alpha}_{R_s,i}\tilde{S}_{e,i} + \sum_{i=1}^{5}\beta_{4i}S_i\right]\\
&+\sum_{i=1}^{3} f_i V_{e,i}^M + \sum_{i=1}^{2} g_i\tilde{V}_{e,i}^M - \sum_{i=1}^{3}\kappa_{3i}V_i^M - \sum_{i=1}^{3}\tilde{\kappa}_{3i}\tilde{V}_i^M + 6C^{(4)}\mu M^M,
\end{aligned}
\tag{3.56}
$$

$$
\begin{aligned}
\mathcal{S}^M = {}& u^M\left[\sum_{i=1}^{3}\alpha_{S,i}S_{e,i} + \sum_{i=1}^{2}\tilde{\alpha}_{S,i}\tilde{S}_{e,i} - \frac{1}{T^2}\nabla_R(Tf_1\zeta^R) + \frac{\mu_n}{T^2}\nabla_R(Tf_3\zeta^R) + \frac{\mu}{T^2}\nabla_R(Tf_2\zeta^R)\right]\\
&+\zeta^M\sum_{i=1}^{5}\frac{\mu\beta_{4i}-\beta_{2i}}{T}S_i + \sum_{i=1}^{3}\frac{\mu\kappa_{3i}-\kappa_{1i}}{T}V_i^M + \sum_{i=1}^{3}\frac{\mu\tilde{\kappa}_{3i}-\tilde{\kappa}_{1i}}{T}\tilde{V}_i^M\\
&+Tg_1\epsilon^{MNRST}V_Nu_R\zeta_S\partial_T\nu_n + Tg_2\epsilon^{MNRST}V_Nu_R\zeta_S\partial_T\nu + 2C_1T\omega^M\\
&-P_K^M\epsilon^{KNRST}\left[\frac{\mu_n}{T}\nabla_N(Tg_1V_Ru_S\xi_T) + \frac{\mu}{T}\nabla_N(Tg_2V_Ru_S\xi_T) - \frac{1}{T}\nabla_N(Tg_3V_Ru_S\xi_T)\right].
\end{aligned}
\tag{3.57}
$$

The scalar $S_5 = T\delta_{\mathcal{B}}\varphi = u^M\xi_M + \mu_n - \mu$ appearing here can be eliminated in favor of $\nabla_M(R_s\xi^M)$ using the Josephson equation. We will like to reiterate that these results are presented in a particular hydrodynamic frame (gained by aligning u^M, T, μ_n, μ along β^μ, Λ_β) and in a "natural" choice of basis for the independent data. They can be transformed to any other preferred hydrodynamic frame or basis by a straight forward substitution.

IV. NULL REDUCTION TO GALILEAN SUPERFLUIDS

We now reduce our null superfluid results to Galilean superfluids. The results are presented in the covariant Newton-Cartan notation and the conventional noncovariant notation (for superfluids coupled to flat spacetime). For more details on the reduction, please refer to [17].

A. Newton-Cartan notation

We start with a quick review of null reduction of null backgrounds to Newton-Cartan backgrounds; for details see [17]. For an excellent review of Newton-Cartan geometries, please refer to the Appendix of [36].

Background and hydrodynamic fields: On our null background $\mathcal{M}_{(d+1)}$, we choose a basis $\{x^M\} = \{x^-, x^\mu\}$ such that the null isometry $\mathcal{V} = \{V = \partial_-, \Lambda_V = 0\}$. The fact that $\mathcal{V}$ is an isometry implies that all the fields in the theory are independent of the x^- coordinate. To perform the reduction, we require an arbitrary null field v^M normalized as $v^Mv_M = 0$, $v^MV_M = -1$, which can be interpreted as providing a "Galilean frame of reference." In the case of a null (super)fluid, the null fluid velocity $v^M = u^M$ defines a special Galilean frame which we refer to as the "fluid frame of reference." In an arbitrary Galilean frame, we decompose the fields V^M, v^M, g_{MN}, A_M in the chosen basis as

$$V^M = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad v^M = \begin{pmatrix} v^\mu B^{(v)}_\mu \\ v^\mu \end{pmatrix},$$
$$g_{MN} = \begin{pmatrix} 0 & -n_\nu \\ -n_\mu & h_{\mu\nu} + 2n_{(\mu}B^{(v)}_{\nu)} \end{pmatrix}, \qquad A_M = \begin{pmatrix} -1 \\ A_\mu \end{pmatrix}, \tag{4.1}$$

along with

$$V_M = \begin{pmatrix} 0 \\ -n_\mu \end{pmatrix}, \qquad v_M = \begin{pmatrix} -1 \\ B^{(v)}_\mu \end{pmatrix},$$
$$g^{MN} = \begin{pmatrix} h^{\nu\rho}B^{(v)}_\nu B^{(v)}_\rho - 2v^\mu B^{(v)}_\mu & h^{\nu\rho}B^{(v)}_\rho - v^\nu \\ h^{\mu\nu}B^{(v)}_\nu - v^\mu & h^{\mu\nu} \end{pmatrix}, \tag{4.2}$$

such that

$$n_\mu v^\mu = 1, \qquad v^\mu h_{\mu\nu} = 0,$$
$$n_\mu h^{\mu\nu} = 0, \qquad h_{\mu\rho}h^{\rho\nu} + n_\mu v^\nu = \delta_\mu{}^\nu. \tag{4.3}$$

The collection of fields $\{n_\mu, v^\mu, h^{\mu\nu}, h_{\mu\nu}, B^{(v)}_\mu\}$ defines a Newton-Cartan structure. The condition $\nabla_M V^N = 0$ implies that the "time-metric" $n = n_\mu \mathrm{d}x^\mu$ is a closed one-form, i.e. $\mathrm{d}n = 0$; this is known to be true for torsionless Newton-Cartan structures. Note that after choosing the said basis, the residual diffeomorphisms are $x^\mu \to x^\mu + \chi^\mu(x^\nu)$ and $x^- \to \xi^- + \chi^-(x^\mu)$. The former of these are just the Newton-Cartan diffeomorphisms, while the latter are known as "mass gauge transformations." Only fields that transform under these mass gauge transformations are,

$$\delta_{\chi^-} B^{(v)}_\mu = -\partial_\mu \chi^-, \qquad \delta_{\chi^-} A_\mu = -\partial_\mu \chi^-. \tag{4.4}$$

$B^{(v)}_\mu$ is therefore known as the mass gauge field. On the other hand mass gauge transformation of A_μ can be absorbed into its U(1) gauge transformation. We define the volume element on a Newton-Cartan background as

$$\varepsilon^{\mu\nu\rho\sigma} = v_M \epsilon^{M\mu\nu\rho\sigma} = -\epsilon^{-\mu\nu\rho\sigma}. \tag{4.5}$$

Note that the volume element is independent of the Galilean frame employed to define it. The Levi-Civita connection Γ^R_{MN} decomposes in this basis as

$$\begin{aligned}\Gamma^\lambda{}_{\mu\nu} &= v^\lambda \partial_{(\mu} n_{\nu)} + \frac{1}{2} h^{\lambda\rho}\left(\partial_\mu h_{\rho\nu} + \partial_\nu h_{\rho\mu} - \partial_\rho h_{\mu\nu}\right) - \Omega^{(v)}_{\sigma(\mu} n_{\nu)} h^{\sigma\lambda}, \\ \Gamma^-{}_{\mu\nu} &= h_{\lambda(\mu}\nabla_{\nu)} v^\lambda - \nabla_{(\mu} B^{(v)}_{\nu)},\end{aligned} \tag{4.6}$$

and all the remaining components zero. Here we have identified $\Gamma^\lambda{}_{\mu\nu}$ as the (torsionless) Newton-Cartan connection and denoted the respective covariant derivative by ∇_μ. We have also defined the (dual) frame vorticity and electromagnetic field strength as

$$\Omega^{(v)}_{\mu\nu} = 2h_{\sigma[\nu}\nabla_{\mu]} v^\sigma = \partial_\mu B^{(v)}_\nu - \partial_\nu B^{(v)}_\mu, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{4.7}$$

The covariant derivative ∇ acts on the Newton-Cartan structure appropriately,

$$\nabla_\mu n_\nu = 0, \qquad \nabla_\mu h^{\rho\sigma} = 0, \qquad \nabla_\mu h_{\nu\rho} = -2n_{(\nu} h_{\rho)\sigma} \nabla_\mu v^\sigma. \tag{4.8}$$

Note that v^M was an arbitrary field chosen to perform the reduction, and one is allowed to arbitrarily redefine it without changing the physics. This leads to the invariance of the system under "Milne transformations" of the Newton-Cartan structure,

$$v^\mu \to v^\mu + \psi^\mu, \qquad h_{\mu\nu} \to h_{\mu\nu} - 2n_{(\mu}\psi_{\nu)} + n_\mu n_\nu \psi^\rho \psi_\rho, \qquad B^{(v)}_\mu \to B^{(v)}_\mu + \psi_\mu - \frac{1}{2} n_\mu \psi^\rho \psi_\rho, \tag{4.9}$$

where $\psi^\mu n_\mu = 0$, $\psi_\mu = h_{\mu\nu}\psi^\nu$. The fields n_μ, $h^{\mu\nu}$, $\Gamma^\rho_{\mu\nu}$ and $\varepsilon^{\mu\nu\rho\sigma}$ are Milne invariant. We can now decompose the fluid velocity u^M and the associated projector P^{MN} as

$$u^M = \begin{pmatrix} u^\mu B_\mu \\ u^\mu \end{pmatrix}, \qquad u_M = \begin{pmatrix} -1 \\ B_\mu \end{pmatrix}, \qquad P_{MN} = \begin{pmatrix} 0 & 0 \\ 0 & p_{\mu\nu} \end{pmatrix}, \qquad P^{MN} = \begin{pmatrix} p^{\nu\rho} B_\nu B_\rho & p^{\mu\nu} B_\nu \\ p^{\mu\nu} B_\nu & p^{\mu\nu} \end{pmatrix}. \tag{4.10}$$

The fields $\{n_\mu, u^\mu, p^{\mu\nu}, p_{\mu\nu}, B_\mu\}$ define the Newton-Cartan structure in the fluid frame of reference, satisfying,

$$n_\mu u^\mu = 1, \qquad u^\mu p_{\mu\nu} = 0, \qquad n_\mu p^{\mu\nu} = 0, \qquad p_{\mu\rho} p^{\rho\nu} + n_\mu u^\nu = \delta^\nu_\mu. \tag{4.11}$$

They can be reexpressed in terms of $\{n_\mu, v^\mu, h^{\mu\nu}, h_{\mu\nu}, B^{(v)}_\mu\}$ using Eq. (4.9) with $\psi^\mu = \bar{u}^\mu = h^\mu{}_\nu u^\nu = u^\mu - v^\mu$,

$$p^{\mu\nu} = h^{\mu\nu}, \qquad p_{\mu\nu} = h_{\mu\nu} - 2n_{(\mu}\bar{u}_{\nu)} + n_\mu n_\nu \bar{u}^\rho \bar{u}_\rho, \qquad B_\mu = B^{(v)}_\mu + \bar{u}_\mu - \frac{1}{2} n_\mu \bar{u}^\rho \bar{u}_\rho. \tag{4.12}$$

The (dual) fluid vorticity is defined similar to the (dual) frame vorticity as

$$\Omega_{\mu\nu} = 2p_{\sigma[\nu} \nabla_{\mu]} u^\sigma = \partial_\mu B_\nu - \partial_\nu B_\mu. \tag{4.13}$$

For later use, we define the magnetic field and fluid vorticity,

$$M^\mu = \frac{1}{2} \epsilon^{\nu\rho\sigma\mu} n_\nu F_{\rho\sigma}, \qquad \omega^\mu = \frac{1}{2} \epsilon^{\nu\rho\sigma\mu} n_\nu \Omega_{\rho\sigma}. \tag{4.14}$$

Finally the superfluid velocity can be decomposed as

$$\zeta^M = \begin{pmatrix} B_\mu \zeta^\mu \\ \zeta^\mu \end{pmatrix}, \qquad \xi^M = \begin{pmatrix} \mu_s + \frac{1}{2} p_{\mu\nu} \zeta^\mu \zeta^\nu + B_\mu \xi^\mu \\ \xi^\mu = \zeta^\mu + u^\mu \end{pmatrix}, \tag{4.15}$$

where $\xi^\mu n_\mu = 1$, $\zeta^\mu n_\mu = 0$. We have treated the superfluid potential μ_s as an independent component of ξ^M. The hatted superfluid potential is, however, given as $\hat{\mu}_s = -\frac{1}{2} \zeta^\mu \zeta_\mu$. Decomposition of the projector $\tilde{P}^{MN}$, on the other hand, is

$$\tilde{P}_{MN} = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{p}_{\mu\nu} = p_{\mu\nu} - \frac{\zeta_\mu \zeta_\nu}{p^{\rho\sigma} \zeta_\rho \zeta_\sigma} \end{pmatrix}, \qquad \tilde{P}^{MN} = \begin{pmatrix} \tilde{p}^{\nu\rho} B_\nu B_\rho & \tilde{p}^{\mu\nu} B_\nu \\ \tilde{p}^{\mu\nu} B_\nu & \tilde{p}^{\mu\nu} = p^{\mu\nu} - \frac{\zeta^\mu \zeta^\nu}{p^{\rho\sigma} \zeta_\rho \zeta_\sigma} \end{pmatrix}. \tag{4.16}$$

Currents and conservation: The mass current ρ^μ, energy current ϵ^μ, stress tensor $t^{\mu\nu}$, charge current j^μ and entropy current s^μ on Newton-Cartan backgrounds can be respectively read out in terms of T^{MN}, J^M, J^M_S as [17],

$$\rho^\mu = -T^{\mu M} V_M, \qquad \epsilon^\mu = -T^{\mu M} u_M, \qquad t^{\mu\nu} = P^\mu_M P^\nu_N T^{MN}, \qquad j^\mu = J^\mu, \qquad s^\mu = J^\mu_S, \tag{4.17}$$

with $t^{\mu\nu} = t^{\nu\mu}$ and $t^{\mu\nu} n_\nu = 0$. They satisfy the conservation laws and the second law of thermodynamics,

$$\begin{aligned} \text{Mass Conservation:} \quad & \nabla_\mu \rho^\mu = 0, \\ \text{Energy Conservation:} \quad & \nabla_\mu \epsilon^\mu = -u^\nu F_{\nu\rho} j^\rho - (u^\mu \rho^\sigma + t^{\mu\sigma}) p_{\sigma\nu} \nabla_\mu u^\nu - \mathrm{T_H}^\perp{}_\nu u^\nu, \\ \text{Momentum Conservation:} \quad & \nabla_\mu (u^\mu p^\sigma{}_\nu \rho^\nu + t^{\mu\sigma}) = p^{\sigma\nu} F_{\nu\rho} j^\rho - \rho^\mu \nabla_\mu u^\sigma + \mathrm{T_H}^\perp{}_\nu p^{\sigma\nu}, \\ \text{Charge Conservation:} \quad & \nabla_\mu j^\mu = \mathrm{J}^\perp_\mathrm{H}, \\ \text{Second Law of Thermo.:} \quad & \nabla_\mu s^\mu \geq 0. \end{aligned} \tag{4.18}$$

The energy current ϵ^μ and the stress tensor $t^{\mu\nu}$ in Eq. (4.17) are defined in the fluid frame of reference; we can define the respective quantities in an arbitrary frame of reference,

$$\begin{aligned} \epsilon^\mu_{(v)} &= -T^{\mu M} v_M = \epsilon^\mu + u^\mu \bar{u}^\nu p_{\nu\rho} \rho^\rho + \frac{1}{2} \rho^\mu \bar{u}^\rho \bar{u}_\rho + t^{\mu\nu} \bar{u}_\nu, \\ t^{\mu\nu}_{(v)} &= (P_{(v)})^\mu_M (P_{(v)})^\nu_N T^{MN} = t^{\mu\nu} + 2\bar{u}^{(\mu} h^{\nu)}_\sigma \rho^\sigma - \bar{u}^\mu \bar{u}^\nu \rho^\sigma n_\sigma, \end{aligned} \tag{4.19}$$

where $P^{MN}_{(v)} = g^{MN} + 2v^{(M}V^{N)}$. They satisfy the conservation laws,

$$
\begin{aligned}
\nabla_\mu \epsilon^\mu_{(v)} &= -v^\nu F_{\nu\rho} j^\rho - \left(v^\mu \rho^\sigma + t^{\mu\sigma}_{(v)}\right) h_{\sigma\nu} \nabla_\mu v^\nu - \mathrm{T_H}{}^\perp{}_\nu v^\nu \\
\nabla_\mu (v^\mu h^\sigma{}_\nu \rho^\nu + t^{\mu\sigma}_{(v)}) &= h^{\sigma\nu} F_{\nu\rho} j^\rho - \rho^\mu \nabla_\mu v^\sigma + \mathrm{T_H}{}^\perp{}_\nu h^{\sigma\nu}.
\end{aligned}
\tag{4.20}
$$

Galilean superfluid constitutive relations: Finally, by a direct computation we can find that the Galilean superfluid constitutive relations in the fluid frame take a structural form.

$$
\begin{aligned}
\rho^\mu &= \rho u^\mu + R_s \xi^\mu + \varsigma^\mu_\rho, \\
\epsilon^\mu &= \epsilon u^\mu + R_s \left(\frac{1}{2}\zeta^\mu \zeta_\mu + \mu_s\right) \xi^\mu + \varsigma^\mu_\epsilon, \\
t^{\mu\nu} &= P p^{\mu\nu} + R_s \zeta^\mu \zeta^\nu + \varsigma^{\mu\nu}_t, \\
j^\mu &= q u^\mu - R_s \xi^\mu + \varsigma^\mu_q, \\
s^\mu &= s u^\mu + \varsigma^\mu_s.
\end{aligned}
\tag{4.21}
$$

While in an arbitrary frame, energy current and stress tensor are given as

$$
\begin{aligned}
\epsilon^\mu_{(v)} &= u^\mu \left(\epsilon + \frac{1}{2}\rho \bar{u}^2 + \varsigma^\sigma_\rho \bar{u}_\sigma\right) + R_s \xi^\mu \left(\frac{1}{2}\bar{\xi}^2 + \mu_s\right) + P\bar{u}^\mu + \left(\varsigma^\mu_\epsilon + \frac{1}{2}\varsigma^\mu_\rho \bar{u}^2 + \varsigma^{\mu\rho}_s \bar{u}_\rho\right), \\
t^{\mu\nu}_{(v)} &= \rho \bar{u}^\mu \bar{u}^\nu + R_s \bar{\xi}^\mu \bar{\xi}^\nu + P h^{\mu\nu} + (\varsigma^{\mu\nu}_s + 2\varsigma^{(\mu}_\rho \bar{u}^{\nu)}),
\end{aligned}
\tag{4.22}
$$

where $\bar{u}^\mu = h^\mu_\nu u^\nu = u^\mu - v^\mu$ and $\bar{\xi}^\mu = h^\mu_\nu \xi^\nu = \xi^\mu - v^\mu$. Various quantities appearing in the constitutive relations can be found via reduction (Table IV) as: fluid densities,

$$
\begin{aligned}
\rho &= R_n + \sum_{i=1}^{3} \alpha_{R_n,i} S_{e,i} + \sum_{i=1}^{2} \tilde{\alpha}_{R_n,i} \tilde{S}_{e,i} - \frac{1}{T}\nabla_\rho (T f_3 \zeta^\rho), \\
\epsilon &= E + \sum_{i=1}^{3} \alpha_{E,i} S_{e,i} + \sum_{i=1}^{2} \tilde{\alpha}_{E,i} \tilde{S}_{e,i} - \frac{1}{T}\nabla_\rho (T f_1 \zeta^\rho) \\
q &= Q + \sum_{i=1}^{3} \alpha_{Q,i} S_{e,i} + \sum_{i=1}^{2} \tilde{\alpha}_{Q,i} \tilde{S}_{q,i} - \frac{1}{T}\nabla_\rho (T f_2 \zeta^\rho), \\
s &= S + \sum_{i=1}^{3} \alpha_{S,i} S_{e,i} + \sum_{i=1}^{2} \tilde{\alpha}_{S,i} \tilde{S}_{e,i} - \frac{1}{T^2}\nabla_\rho (T f_1 \zeta^\rho) + \frac{\mu_n}{T^2}\nabla_\rho (T f_3 \zeta^\rho) + \frac{\mu}{T^2}\nabla_\rho (T f_2 \zeta^\rho).
\end{aligned}
\tag{4.23}
$$

and dissipative currents,

$$
\begin{aligned}
\varsigma^\mu_\rho &= \zeta^\mu \left[\sum_{i=1}^{3} \alpha_{R_s,i} S_{e,i} + \sum_{i=1}^{2} \tilde{\alpha}_{R_s,i} \tilde{S}_{e,i}\right] - \sum_{i=1}^{3} f_i V^\mu_{e,i} - \sum_{i=1}^{2} g_i \tilde{V}^\mu_{e,i} + \varepsilon^{\mu\nu\rho\sigma} \partial_\nu (T g_1 n_\rho \zeta_\sigma), \\
\varsigma^\mu_\epsilon &= \zeta^\mu \left[\sum_{i=1}^{3} f_i S_{5+i} + \left(\mu_s + \frac{1}{2}\zeta^\mu \zeta_\mu\right)\left(\sum_{i=1}^{3} \alpha_{R_s,i} S_{e,i} + \sum_{i=1}^{2} \tilde{\alpha}_{R_s,i} \tilde{S}_{e,i}\right) - \sum_{i=1}^{5} \beta_{2i} S_i\right] \\
&\quad + \left(\mu_s + \frac{1}{2}\zeta^\mu \zeta_\mu\right) \sum_{i=1}^{3} f_i V^\mu_{e,i} - \sum_{i=1}^{2} g_i \tilde{V}^\mu_{e,i+2} - \sum_{i=1}^{3} \kappa_{1i} V^\mu_i - \sum_{i=1}^{3} \tilde{\kappa}_{1i} \tilde{V}^\mu_i + 3C^{(4)} \mu^2 M^\mu \\
&\quad + \varepsilon^{\mu\nu\rho\sigma} \partial_\nu (T g_3 n_\rho \zeta_\sigma) + C_1 T^2 \omega^\mu,
\end{aligned}
$$

$$
\begin{aligned}
\varsigma_t^{\mu\nu} &= \zeta^\mu\zeta^\nu\left[\sum_{i=1}^{3}\alpha_{R_s,i}S_{e,i} + \sum_{i=1}^{2}\tilde{\alpha}_{R_s,i}\tilde{S}_{e,i} - \sum_{i=1}^{2}\frac{g_i}{2\hat{\mu}_s}\tilde{S}_{e,i} - \sum_{i=1}^{5}\beta_{3i}S_i\right] - \eta\sigma^{\mu\nu} - \tilde{\eta}\tilde{\sigma}^{\mu\nu} \\
&\quad - 2\zeta^{(\mu}\left[\sum_{i=1}^{3}f_i V_{e,i}^{\nu)} + \sum_{i=1}^{3}\kappa_{2i}V_i^{\nu)} + \sum_{i=1}^{3}\tilde{\kappa}_{2i}\tilde{V}_i^{\nu)}\right] + \tilde{p}^{\mu\nu}\left[\sum_{i=1}^{3}f_i S_{e,i} - \sum_{i=1}^{5}\beta_{1i}S_i\right], \\
\varsigma_q^{\mu} &= -\zeta^\mu\left[\sum_{i=1}^{3}\alpha_{R_s,i}S_{e,i} + \sum_{i=1}^{2}\tilde{\alpha}_{R_s,i}\tilde{S}_{e,i} + \sum_{i=1}^{5}\beta_{4i}S_i\right] + \epsilon^{\mu\nu\rho\sigma}\partial_\nu(Tg_2 n_\rho\zeta_\sigma) \\
&\quad + \sum_{i=1}^{3}f_i V_{e,i}^{\mu} + \sum_{i=1}^{2}g_i\tilde{V}_{e,i}^{\mu} - \sum_{i=1}^{3}\kappa_{3i}V_i^\mu - \sum_{i=1}^{3}\tilde{\kappa}_{3i}\tilde{V}_i^\mu + 6C^{(4)}\mu M^\mu, \\
\varsigma_s^{\mu} &= \zeta^\mu\sum_{i=1}^{5}\frac{\mu\beta_{4i}-\beta_{2i}}{T}S_i - \epsilon^{\mu\nu\rho\sigma}\left[\frac{\mu_n}{T}\partial_\nu(Tg_1 n_\rho\zeta_\sigma) + \frac{\mu}{T}\partial_\nu(Tg_2 n_\rho\zeta_\sigma) - \frac{1}{T}\partial_\nu(Tg_3 n_\rho\zeta_\sigma)\right] \\
&\quad + \sum_{i=1}^{3}\frac{\mu\kappa_{3i}-\kappa_{1i}}{T}V_i^\mu + \sum_{i=1}^{3}\frac{\mu\tilde{\kappa}_{3i}-\tilde{\kappa}_{1i}}{T}\tilde{V}_i^\mu - Tg_1\epsilon^{\mu\nu\rho\sigma}n_\nu\zeta_\rho\partial_\sigma\nu_n - Tg_2\epsilon^{\mu\nu\rho\sigma}n_\nu\zeta_\rho\partial_\sigma\nu \\
&\quad + 2C_1 T\omega^\mu.
\end{aligned}
\tag{4.24}
$$

In addition, we also have the Josephson equation,

$$
\begin{aligned}
-\frac{1}{2}\zeta^\mu\zeta_\mu - \mu_s + \mu_n - \mu &= \frac{1}{\beta_{55}}\nabla_\mu(R_s\xi^\mu) - \sum_{i=1}^{4}\frac{\beta_{5i}}{\beta_{55}}S_i \\
&\quad + \frac{1}{\beta_{55}}\nabla_\mu\left(\zeta^\mu\sum_{i=1}^{3}\alpha_{R_s,i}S_{e,i} + \zeta^\mu\sum_{i=1}^{2}\tilde{\alpha}_{R_s,i}\tilde{S}_{e,i} - \sum_{i=1}^{3}f_i V_{e,i}^\mu - \sum_{i=1}^{2}g_i\tilde{V}_{e,i}^\mu\right),
\end{aligned}
\tag{4.25}
$$

which is the derivative correction of the ideal order version $\mu_s = -\frac{1}{2}\zeta^\mu\zeta_\mu - \mu + \mu_n$. This completes our discussion of the first-order Galilean (Newton-Cartan) superfluids; counting of various transport coefficients appearing in the constitutive relations is same as the null superfluid given in Sec. III E.

B. Noncovariant notation (for flat spacetime)

If the superfluid is coupled to a flat Galilean spacetime, it is fitting to reexpress the results in the conventional noncovariant notation where we treat the time and space indices distinctly. It might help the reader to better relate the Galilean superfluid constitutive relations to the existing Galilean literature, e.g. in [23].

Background and hydrodynamic fields: On the Newton-Cartan background, we choose a basis $\{x^\mu\} = \{t, x^i\}$ such that the Galilean frame velocity $(v^\mu) = \partial_t$. A flat Galilean background is defined by a particular choice of the Newton-Cartan structure in this basis,

$$
n_\mu = \begin{pmatrix}1\\0\end{pmatrix}, \qquad v^\mu = \begin{pmatrix}1\\0\end{pmatrix}, \qquad p^{\mu\nu} = \begin{pmatrix}0 & 0\\0 & \delta^{ij}\end{pmatrix}, \qquad p_{\mu\nu} = \begin{pmatrix}0 & 0\\0 & \delta_{ij}\end{pmatrix}, \qquad B_\mu^{(v)} = 0, \tag{4.26}
$$

where $\delta^{ij} = \delta_{ij}$ is the Kronecker delta. It can be checked that the respective Newton-Cartan connection $\Gamma^\lambda_{\mu\nu} = 0$, justifying the spacetime to be flat. The Newton-Cartan structure in the fluid frame can be worked out from here to be,

$$
u^\mu = \begin{pmatrix}1\\u^i\end{pmatrix}, \qquad B_\mu = \begin{pmatrix}-\frac{1}{2}u^k u_k\\u_i\end{pmatrix}, \qquad p^{\mu\nu} = \begin{pmatrix}0 & 0\\0 & \delta^{ij}\end{pmatrix}, \qquad p_{\mu\nu} = \begin{pmatrix}u^k u_k & -u_j\\-u_i & \delta_{ij}\end{pmatrix}. \tag{4.27}
$$

We define the spatial volume element,

$$
\epsilon^{ijk} = n_\mu\epsilon^{\mu ijk} = \epsilon^{tijk}. \tag{4.28}
$$

The U(1) gauge field A_μ can be decomposed as $A_\mu \mathrm{d}x^\mu = A_t \mathrm{d}t + A_i \mathrm{d}x^i$. The fluid vorticity and electromagnetic field strength on the other hand can be decomposed as

$$\Omega_{\mu\nu} = \begin{pmatrix} 0 & (\partial_t + u^k\partial_k)u_i + \omega_{ik}u^k \\ -(\partial_t + u^k\partial_k)u_i - \omega_{ik}u^k & \omega_{ij} = \partial_i u_j - \partial_j u_i \end{pmatrix}, \tag{4.29}$$

$$F_{\mu\nu} = \begin{pmatrix} 0 & -e_i = \partial_t A_i - \partial_i A_t \\ e_i = -\partial_t A_i + \partial_i A_t & \beta_{ij} = \partial_i A_j - \partial_j A_i \end{pmatrix}, \tag{4.30}$$

where ω_{ij} is the (dual) spatial vorticity, e_i is the electric field and β_{ij} is the dual magnetic field. For later use, we define the magnetic field and fluid vorticity,

$$M^i = \frac{1}{2}\varepsilon^{ijk}\beta_{jk}, \qquad \omega^i = \frac{1}{2}\varepsilon^{ijk}\omega_{jk}. \tag{4.31}$$

Finally the superfluid velocity can be decomposed as

$$\zeta^\mu = \begin{pmatrix} 0 \\ \zeta^i \end{pmatrix}, \qquad \xi^\mu = \begin{pmatrix} 1 \\ \xi^i = u^i + \zeta^i \end{pmatrix},$$
$$\mu_s = -\xi_t - \frac{1}{2}\xi^i\xi_i, \qquad \hat{\mu}_s = -\frac{1}{2}\zeta^i\zeta_i, \tag{4.32}$$

with the projection operators,

$$\tilde{p}_{\mu\nu} = \begin{pmatrix} u^k u_k & -u_j \\ -u_i & \tilde{p}_{ij} = \delta_{ij} - \frac{\zeta_i\zeta_j}{\zeta^k\zeta_k} \end{pmatrix},$$
$$\tilde{p}^{\mu\nu} = \begin{pmatrix} 0 & 0 \\ 0 & \tilde{p}^{ij} = \delta^{ij} - \frac{\zeta^i\zeta^j}{\zeta^k\zeta_k} \end{pmatrix}. \tag{4.33}$$

Densities, currents and conservation laws: In flat spacetime, the conservation laws and the second law of thermodynamics take the well-known form,

$$\begin{aligned} \text{mass conservation}&: \partial_t\rho^t + \partial_i\rho^i = 0 \\ \text{energy conservation}&: \partial_t\epsilon^t_{(v)} + \partial_i\epsilon^i_{(v)} = j^i e_i - \mathrm{T}_{\mathrm{H}^\perp_t} \\ \text{momentum conservation}&: \partial_t\rho^j + \partial_i t^{ij}_{(v)} = (e^j j^t + \beta^{jk} j_k) + \mathrm{T_H}^{\perp j}, \\ \text{charge conservation}&: \partial_t j^t + \partial_i j^i = \mathrm{J}^\perp_{\mathrm{H}}, \\ \text{second law of thermodynamics}&: \partial_t s^t + \partial_i s^i \geq 0, \end{aligned} \tag{4.34}$$

where we have identified various Galilean quantities: mass density ρ^t, mass current ρ^i, energy density $\epsilon^t_{(v)}$, energy current $\epsilon^i_{(v)}$, stress tensor $t^{ij}_{(v)}$, charge density j^t, charge current j^i, entropy density s^t and entropy current s^i.

Superfluid constitutive relations: Finally, we can read out the structural form of the Galilean superfluid constitutive relations in noncovariant notation using reduction,

$$\begin{aligned} \rho^t &= \rho + R_s, \qquad \rho^i = \rho u^i + R_s\xi^i + \varsigma^i_\rho, \\ \epsilon^t_{(v)} &= \epsilon + R_s\mu_s + \frac{1}{2}\rho\vec{u}^2 + \frac{1}{2}R_s\vec{\xi}^2 + \varsigma^i_\rho u_i, \\ \epsilon^i_{(v)} &= u^i\left(\epsilon + P + \frac{1}{2}\rho\vec{u}^2 + \varsigma^j_\rho u_j\right) + R_s\xi^i\left(\frac{1}{2}\vec{\xi}^2 + \mu_s\right) + \left(\varsigma^i_\epsilon + \frac{1}{2}\varsigma^i_\rho\vec{u}^2 + \varsigma^{ij}_s u_j\right), \\ t^{ij}_{(v)} &= \rho u^i u^j + R_s\xi^i\xi^j + P\delta^{ij} + (\varsigma^{ij}_s + 2\varsigma^{(i}_\rho u^{j)}), \\ j^t &= q - R_s, \qquad j^i = qu^i - R_s\xi^i + \varsigma^i_q, \\ s^t &= s, \qquad s^i = su^i + \varsigma^i_s. \end{aligned} \tag{4.35}$$

Various quantities appearing here can also be worked out using reduction: fluid densities,

$$\rho = R_n + \sum_{i=1}^{3}\alpha_{R_n,i}S_{e,i} + \sum_{i=1}^{2}\tilde{\alpha}_{R_n,i}\tilde{S}_{e,i} - \frac{1}{T}\partial_i(Tf_3\zeta^i),$$
$$\epsilon = E + \sum_{i=1}^{3}\alpha_{E,i}S_{e,i} + \sum_{i=1}^{2}\tilde{\alpha}_{E,i}\tilde{S}_{e,i} - \frac{1}{T}\partial_i(Tf_1\zeta^i)$$

$$q = Q + \sum_{i=1}^{3} \alpha_{Q,i} S_{e,i} + \sum_{i=1}^{2} \tilde{\alpha}_{Q,i} \tilde{S}_{q,i} - \frac{1}{T} \partial_i (T f_2 \zeta^i),$$
$$s = S + \sum_{i=1}^{3} \alpha_{S,i} S_{e,i} + \sum_{i=1}^{2} \tilde{\alpha}_{S,i} \tilde{S}_{e,i} - \frac{1}{T^2} \partial_i (T f_1 \zeta^i) + \frac{\mu_n}{T^2} \partial_i (T f_3 \zeta^i) + \frac{\mu}{T^2} \partial_i (T f_2 \zeta^i), \tag{4.36}$$

and dissipative currents,

$$\varsigma_\rho^i = \zeta^i \left[\sum_{i=1}^{3} \alpha_{R_s,i} S_{e,i} + \sum_{i=1}^{2} \tilde{\alpha}_{R_s,i} \tilde{S}_{e,i} \right] - \sum_{i=1}^{3} f_i V_{e,i}^i - \sum_{i=1}^{2} g_i \tilde{V}_{e,i}^i + \varepsilon^{ijk} \partial_j (T g_1 \zeta_k),$$
$$\varsigma_\epsilon^i = \zeta^i \left[\sum_{i=1}^{3} f_i S_{5+i} + \left(\mu_s + \frac{1}{2} \zeta^k \zeta_k \right) \left(\sum_{i=1}^{3} \alpha_{R_s,i} S_{e,i} + \sum_{i=1}^{2} \tilde{\alpha}_{R_s,i} \tilde{S}_{e,i} \right) - \sum_{i=1}^{5} \beta_{2i} S_i \right]$$
$$+ \left(\mu_s + \frac{1}{2} \zeta^k \zeta_k \right) \sum_{i=1}^{3} f_i V_{e,i}^i - \sum_{i=1}^{2} g_i \tilde{V}_{e,i+2}^i - \sum_{i=1}^{3} \kappa_{1i} V_i^i - \sum_{i=1}^{3} \tilde{\kappa}_{1i} \tilde{V}_i^i + 3C^{(4)} \mu^2 M^i,$$
$$+ \varepsilon^{ijk} \partial_j (T g_3 \zeta_k) + C_1 T^2 \omega^i$$
$$\varsigma_t^{ij} = \zeta^i \zeta^j \left[\sum_{i=1}^{3} \alpha_{R_s,i} S_{e,i} + \sum_{i=1}^{2} \tilde{\alpha}_{R_s,i} \tilde{S}_{e,i} - \sum_{i=1}^{2} \frac{g_i}{2\hat{\mu}_s} \tilde{S}_{e,i} - \sum_{i=1}^{5} \beta_{3i} S_i \right] - \eta \sigma^{ij} - \tilde{\eta} \tilde{\sigma}^{ij}$$
$$- 2\zeta^{(i} \left[\sum_{i=1}^{3} f_i V_{e,i}^{j)} + \sum_{i=1}^{3} \kappa_{2i} V_i^{j)} + \sum_{i=1}^{3} \tilde{\kappa}_{2i} \tilde{V}_i^{j)} \right] + \tilde{p}^{ij} \left[\sum_{i=1}^{3} f_i S_{e,i} - \sum_{i=1}^{5} \beta_{1i} S_i \right],$$
$$\varsigma_q^i = -\zeta^i \left[\sum_{i=1}^{3} \alpha_{R_s,i} S_{e,i} + \sum_{i=1}^{2} \tilde{\alpha}_{R_s,i} \tilde{S}_{e,i} + \sum_{i=1}^{5} \beta_{4i} S_i \right] + \varepsilon^{ijk} \partial_j (T g_2 \zeta_k)$$
$$+ \sum_{i=1}^{3} f_i V_{e,i}^i + \sum_{i=1}^{2} g_i \tilde{V}_{e,i}^i - \sum_{i=1}^{3} \kappa_{3i} V_i^i - \sum_{i=1}^{3} \tilde{\kappa}_{3i} \tilde{V}_i^i + 6C^{(4)} \mu M^i,$$
$$\varsigma_s^i = \zeta^i \sum_{i=1}^{5} \frac{\mu \beta_{4i} - \beta_{2i}}{T} S_i - \varepsilon^{ijk} \left[\frac{\mu_n}{T} \partial_j (T g_1 \zeta_k) + \frac{\mu}{T} \partial_j (T g_2 \zeta_k) - \frac{1}{T} \partial_j (T g_3 \zeta_k) \right]$$
$$+ \sum_{i=1}^{3} \frac{\mu \kappa_{3i} - \kappa_{1i}}{T} V_i^i + \sum_{i=1}^{3} \frac{\mu \tilde{\kappa}_{3i} - \tilde{\kappa}_{1i}}{T} \tilde{V}_i^i + T g_1 \varepsilon^{ijk} \zeta_j \partial_k \nu_n + T g_2 \varepsilon^{ijk} \zeta_j \partial_k \nu + 2C_1 T \omega^i. \tag{4.37}$$

In addition, we have the Josephson equation,

$$-\frac{1}{2} \zeta^i \zeta_i - \mu_s + \mu_n - \mu = \frac{1}{\beta_{55}} (\partial_t R_s + \partial_i (R_s \xi^i)) - \sum_{i=1}^{4} \frac{\beta_{5i}}{\beta_{55}} S_i$$
$$+ \frac{1}{\beta_{55}} \partial_k \left(\zeta^k \sum_{i=1}^{3} \alpha_{R_s,i} S_{e,i} + \zeta^k \sum_{i=1}^{2} \tilde{\alpha}_{R_s,i} \tilde{S}_{e,i} - \sum_{i=1}^{3} f_i V_{e,i}^k - \sum_{i=1}^{2} g_i \tilde{V}_{e,i}^k \right), \tag{4.38}$$

which is the derivative correction of the ideal-order version $\mu_s = -\frac{1}{2} \zeta^i \zeta_i + \mu_n - \mu$. These equation can be compared with [23] for which the U(1) chemical potential $\mu = 0$. This completes our discussion of Galilean superfluids coupled to flat Galilean spacetime, expressed in noncovariant notation.

V. GALILEAN (SUPER)FLUIDS VIA LARGE C LIMIT

In the preceding sections, building upon our previous work in [17,18,22], we have formulated a theory of Galilean (super) fluids, illustrated with an explicit computation up to first order in derivatives. The analysis has been purely based on Galilean symmetries and the second law of thermodynamics. However, we know that the nature is fundamentally relativistic, and the physical systems behave Galilean only in the limit $|\vec{v}| \ll c$. It is natural to ask therefore, that to what degree can we trust our construction of "Galilean" (super)fluids to describe "nonrelativistic" physics we encounter around us. The question is particularly important as the number of transport coefficients in a Galilean (super) fluid at a given derivative order, are much more than in a relativistic (super)fluid (see Table I). To bridge this gap between nonrelativistic and Galilean (super)fluids, we would like to be able to show that the most generic Galilean (super)fluid can be gained via an appropriate $c \to \infty$ limit of a relativistic system.

An important point to note here is that we do not require this "relativistic system", whose limit leads to the Galilean fluid, to be a "relativistic fluid". The reason is that fluid dynamics itself is an effective theory of large wavelength fluctuations, and there is no reason to expect that the $c \to \infty$ and large wavelength limits would commute (see [37] for more discussion in this direction). In other words, there might be some information in the microscopic field theory which gets integrated over in the long wavelength limit, but nevertheless survives a $c \to \infty$ limit followed by a long wavelength limit. In fact, to be able to take the $c \to \infty$ limit consistently, the relativistic fluid needs to be accompanied with an additional U(1) current which keeps track of the flow of mass. This requirement follows from the fact that the nonrelativistic symmetry algebra has an additional mass generator compared to the Poincaré algebra. For the cases where the relativistic fluid comes with a predefined notion of "particle(s)," this current can be provided by the particle number currents, as illustrated by [38] for single component fluids. Practically, this amounts to starting from a relativistic "fluid" with two U(1) currents, one for electromagnetic charge and the other for mass conservation. It is this extra information which leads to more transport coefficients in a Galilean (super)fluid. To get some intuition of this extra information, note that for a single component Galilean (super)fluid, wherein the charge and mass currents are proportional (see footnote 9), the number of transport coefficients turn out to be the same as a relativistic (super)fluid. This suggests that the extra information in a Galilean (super)fluid can be attributed to the presence of multiple components with different charge is to mass ratios. In these fluids mass flows independent of charge, which a relativistic description cannot probe, but is captured in a nonrelativistic description.

To be more concrete, consider the constitutive relations of a relativistic superfluid $T^{\mu\nu}_{\rm rel}$, $J^\mu_{\rm rel}$, $K_{\rm rel}$ and the entropy current $J^\mu_{S,\rm rel}$, written in terms of the fields $u^\mu_{\rm rel}$, $T_{\rm rel}$, $\mu_{\rm rel}$ and $\varphi_{\rm rel}$, and the background fields $g^{\rm rel}_{\mu\nu}$ and $A^{\rm rel}_\mu$. As discussed in Sec. II, they are the most generic expressions allowed by symmetries which satisfy the off-shell second law of thermodynamics,

$$T_{\rm rel}\nabla^{\rm rel}_\mu J^\mu_{S,\rm rel} + u^{\rm rel}_\nu(\nabla^{\rm rel}_\mu T^{\mu\nu}_{\rm rel} - F^{\nu\rho}_{\rm rel}J^{\rm rel}_\rho - \mathrm{T}^{\perp\nu}_{\rm H} - K_{\rm rel}\xi^\nu_{\rm rel}) + \mu_{\rm rel}(\nabla^{\rm rel}_\mu J^\mu_{\rm rel} - \mathrm{J}^\perp_{\rm H} + K_{\rm rel}) \geq 0. \tag{5.1}$$

We depart from this fluid slightly by introducing another U(1) conserved current $R^\mu_{\rm rel}$, along with an associated chemical potential $\mu^{\rm rel}_n$ and a background gauge field $B^{\rm rel}_\mu$. This extended system will be required to satisfy a modified second law,

$$\begin{aligned} &T_{\rm rel}\nabla^{\rm rel}_\mu J^\mu_{S,\rm rel} \\ &\quad + u^{\rm rel}_\nu(\nabla^{\rm rel}_\mu T^{\mu\nu}_{\rm rel} - F^{\nu\rho}_{\rm rel}J^{\rm rel}_\rho - \Omega^{\nu\rho}_{\rm rel}R^{\rm rel}_\rho - \mathrm{T}^{\perp\nu}_{\rm H} - K_{\rm rel}\xi^\nu_{\rm rel}) \\ &\quad + \mu_{\rm rel}(\nabla^{\rm rel}_\mu J^\mu_{\rm rel} - \mathrm{J}^\perp_{\rm H} + K_{\rm rel}) + \mu^{\rm rel}_n(\nabla^{\rm rel}_\mu R^\mu_{\rm rel} - K_{\rm rel}) \geq 0, \end{aligned} \tag{5.2}$$

with $\Omega^{\rm rel}_{\mu\nu} = \partial_\mu B^{\rm rel}_\nu - \partial_\nu B^{\rm rel}_\mu$. Here the superfluid phase φ transforms under both the U(1)'s and the superfluid velocity is given by $\xi^{\rm rel}_\mu = \partial_\mu\varphi^{\rm rel} + A^{\rm rel}_\mu - B^{\rm rel}_\mu$. We claim that under an appropriate $c \to \infty$ limit this system gives rise to the most generic Galilean superfluid. Unlike $T^{\mu\nu}_{\rm rel}$ and $J^\mu_{\rm rel}$ which are associated with fundamental symmetries, not every relativistic system need to have a conserved $R^\mu_{\rm rel}$; it corresponds to an emergent U(1) symmetry at nonrelativistic scales, such as the particle number conservation, which is required to be able to take a nonrelativistic limit consistently.[9]

Let us now proceed to define a $c \to \infty$ limit of this system. For simplicity, we will work in a frame locally comoving with the fluid; results in any other frame can be obtained easily by performing a Galilean boost. We define c-scaling of the background fields as

$$\begin{aligned} &g^{\rm rel}_{\mu\nu} = -c^2 n_\mu n_\nu + p_{\mu\nu}, \qquad g^{\mu\nu}_{\rm rel} = -\frac{1}{c^2}u^\mu u^\nu + p^{\mu\nu}, \\ &A^{\rm rel}_\mu = A_\mu, \qquad B^{\rm rel}_\mu = c^2 n_\mu + B_\mu. \end{aligned} \tag{5.4}$$

[9]One way to interpret such a relativistic system is to consider a fluid with multiple "components" individually conserved. The corresponding currents are then $T^{\mu\nu}_{\rm rel}$, $N^\mu_{a,\rm rel}$, $J^\mu_{S,\rm rel}$ where index "a" runs over the number of components. If each component has rest-mass m_a and charge q_a (normalized such that $\sum_a q_a q_a = -\sum_a m_a q_a = 1$) we can define $J^\mu_{\rm rel} = \sum_a q_a N^\mu_{a,\rm rel}$ and $R^\mu_{\rm rel} = \sum_a m_a N^\mu_{a,\rm rel}$. We turn on a background gauge field $A^{a,\rm rel}_\mu = q^a A^{\rm rel}_\mu + m^a B^{\rm rel}_\mu$ coupling to component currents with respective chemical potentials $\mu^{\rm rel}_a = q^a\mu^{\rm rel} + m^a\mu^{\rm rel}_n$. On the other hand, superfluid velocity is given via $\xi^{\rm rel}_\mu = \partial_\mu\varphi + \sum_a q_a A^a_\mu$. Now the off-shell second law in Eq. (5.2) follows from the off-shell second law of this multicomponent fluid ($\bar{q}_a = (m^2 q_a + m_a)/(\sum_a m_a m_a - 1)$),

$$\begin{aligned} &T_{\rm rel}\nabla^{\rm rel}_\mu J^\mu_{S,\rm rel} + u^{\rm rel}_\nu(\nabla^{\rm rel}_\mu T^{\mu\nu}_{\rm rel} - \sum_a F^{\nu\rho}_{a,\rm rel}J^{a,\rm rel}_\rho - \mathrm{T}^{\perp\nu}_{\rm H} - K_{\rm rel}\xi^\nu_{\rm rel}) \\ &\quad + \sum_a \mu^{\rm rel}_a(\nabla^{\rm rel}_\mu J^\mu_{a,\rm rel} - \bar{q}_a\mathrm{J}^\perp_{\rm H} + q_a K_{\rm rel}) \geq 0. \end{aligned} \tag{5.3}$$

See Sec. IV for the definition of Newton-Cartan fields used here. On the other hand, superfluid fields in a comoving frame scale as

$$u^\mu_{\rm rel} = u^\mu, \qquad T_{\rm rel} = T, \qquad \mu_{\rm rel} = \mu, \qquad \mu_n^{\rm rel} = c^2 + \mu_n,$$
$$\xi^\mu_{\rm rel} = u^\mu \sqrt{1 + \frac{2\mu_s + \zeta^2}{c^2}} + \zeta^\mu. \tag{5.5}$$

In terms of these, we can define various nonrelativistic currents as

$$\rho^\mu = \lim_{c\to\infty} R^\mu_{\rm rel}, \qquad \epsilon^\mu = \lim_{c\to\infty} c^2 (T^{\mu\nu}_{\rm rel} n_\nu - R^\mu_{\rm rel}),$$
$$t^{\mu\nu} = \lim_{c\to\infty} p^\mu_\rho p^\nu_\sigma T^{\rho\sigma}_{\rm rel},$$
$$q^\mu = \lim_{c\to\infty} J^\mu_{\rm rel}, \qquad s^\mu = \lim_{c\to\infty} J^\mu_{S,\rm rel}. \tag{5.6}$$

It can be checked that under a Poincaré transformation of the relativistic currents, these nonrelativistic currents transform appropriately under the Galilean symmetry group. They also satisfy the $c \to \infty$ version of the modified off-shell second law (5.2),

$$T\nabla_\mu s^\mu - \left[\nabla_\mu \epsilon^\mu + (u^\mu \rho^\sigma + t^{\mu\sigma}) p_{\sigma\nu} \nabla_\mu u^\nu + u^\mu F_{\mu\nu} q^\nu + u^\mu \mathrm{T}^\perp_{\mathrm{H}\mu} - K\left(\mu_s + \frac{1}{2}\zeta^2\right)\right]$$
$$+ \mu\left(\nabla_\mu q^\mu - \mathrm{J}^\perp_{\mathrm{H}} + K\right) + \mu_n (\nabla_\mu \rho^\mu - K) + \mathcal{O}(1/c^2) \geq 0, \tag{5.7}$$

which is the correct Galilean off-shell second law. In this way, we can verify that corresponding to every set of Galilean superfluid constitutive relations that satisfy Eq. (5.7), there exists a relativistic system (not necessarily a fluid) satisfying Eq. (5.2) whose $c \to \infty$ limit reduces to the said Galilean superfluid. In other words, every Galilean superfluid is nonrelativistic, i.e.; it follows from the $c \to \infty$ limit of a relativistic system.

VI. DISCUSSION

We worked out the most generic constitutive relations of an (anomalous) Galilean superfluid up to first order in derivative expansion, both in parity-even and -odd sectors. We extended the idea of null fluid introduced in [17,18] to null superfluid, which is a relativistic embedding of a Galilean superfluid in one higher dimension, and used it to obtain the mentioned results. We found the spectrum of transport coefficients to be extremely rich with 38 coefficients in parity-even and 13 coefficients in parity-odd sector at first order, in addition to two undetermined constants in parity-odd sector including the U(1) anomaly constant (see Table I). Out of these, 3 parity-odd and 3 parity-even coefficients survive in equilibrium and determine the hydrostatic physics, while 13 parity-even and 7 parity-odd coefficients govern nondissipative phenomenon away from equilibrium. On the other hand, 22 parity-even and 3 parity-odd coefficients are dissipative. Though we did not discuss it in the main text, there are hints that 13 parity-even nondissipative nonhydrostatic coefficients and 3 parity-odd dissipative coefficients vanish on imposing Onsager relations (microscopic reversibility). To avoid confusion with counting, we would like to note that we have removed one parity-even hydrostatic coefficient by redefinition of the U(1) phase φ.

Perhaps the most striking benefit of working in the off-shell formalism is that it leads to a complete classification of (super)fluid transport up to all orders in derivative expansion [25–27] and provides a natural setting to attempt writing down a Wilsonian effective action describing the entire (super)fluid dynamics [26,39–44]. It will be interesting to undertake these ambitious problems in context of null/Galilean (super)fluids, and we plan to return to these in future.

In this paper, we focused on breaking the internal U(1) symmetry of Galilean fluids and obtain a null/Galilean superfluid. The same procedure can also be used to break spacetime symmetries, which lead to the formation of boundaries/surfaces in (super)fluids [45]. In an upcoming paper [46], authors discuss the surface transport for relativistic and Galilean superfluids. Finally, first-order computations of this paper can also be easily extended to higher orders; in an ongoing project [47] we are looking at some interesting second-order phenomenon in Galilean (super)fluids.

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APPENDIX A: RELATIVISTIC SUPERFLUIDS UP TO FIRST ORDER: DERIVATION

In this Appendix, we present a detailed derivation of first-order constitutive relations of a relativistic superfluid in off-shell formalism. These results have already been obtained in on-shell formalism in [12–14], while a generic mechanism for arbitrarily high derivative order (non-Abelian) superfluids was presented in [27].

1. Ideal superfluids

Let us start with ideal superfluids, i.e. superfluid constitutive relations that satisfy the free energy conservation Eq. (2.16) at first derivative order. At ideal order, the most generic tensorial form of various quantities appearing in Eq. (2.16) can be written as

$$\begin{aligned} T^{\mu\nu} &= (E+P)u^\mu u^\nu + P g^{\mu\nu} + R_s \xi^\mu \xi^\nu + \lambda(u^\mu \xi^\nu + u^\nu \xi^\mu) \\ &\quad + \mathcal{O}(\partial), \\ J^\mu &= Q u^\mu + Q_s \xi^\mu + \mathcal{O}(\partial), \\ K &= -\alpha \delta_{\mathcal{B}}\varphi + K_{\text{ideal}} + \mathcal{O}(\partial), \\ N^\mu &= N u^\mu + N_s \xi^\mu + \mathcal{O}(\partial), \\ \Delta &= (\alpha\delta_{\mathcal{B}}\varphi)^2 + \Delta_{\text{ideal}} + \mathcal{O}(\partial^2), \end{aligned} \tag{A1}$$

where $E, P, R_s, \lambda, Q, Q_s, K_{\text{ideal}}, N, N_s$ are functions of T, μ and $\mu_s \equiv -\frac{1}{2}\xi^\mu \xi_\mu$. We have omitted the only other possible scalar $\delta_{\mathcal{B}}\varphi$ in the functional dependence, because using the φ equation of motion we know that it is no longer an independent quantity. Plugging Eq. (A1) in Eq. (2.16) we can find,

$$\begin{aligned} &(Q_s + R_s)\xi^\mu \left(\nabla_\mu \nu + \frac{1}{T} u^\nu F_{\nu\mu} \right) \\ &\quad + \lambda \xi^\mu \left(\frac{1}{T^2} \nabla_\mu T + u^\nu \nabla_\nu \left(\frac{u_\mu}{T} \right) \right) + \nabla_\mu \left(\left(\frac{P}{T} - N \right) u^\mu \right) \\ &\quad + \frac{1}{T} u^\mu (\nabla_\mu E - T\nabla_\mu S - \mu \nabla_\mu Q + R_s \nabla_\mu \mu_s) \\ &\quad + \nabla_\mu (\delta_{\mathcal{B}}\varphi R_s \xi^\mu - N_s \xi^\mu) \\ &\quad + (K_{\text{ideal}} - \nabla_\mu (R_s \xi^\mu))\delta_{\mathcal{B}}\varphi + \Delta_{\text{ideal}} = 0, \end{aligned} \tag{A2}$$

where we have defined S through the "Euler equation,"

$$E + P = ST + Q\mu. \tag{A3}$$

Equation (A2) will imply a set of relations among various coefficients,

$$\begin{aligned} &Q_s = -R_s, \qquad \lambda = 0, \qquad N = \frac{P}{T}, \qquad N_s = \delta_{\mathcal{B}}\varphi R_s, \\ &K_{\text{ideal}} = \nabla_\mu (R_s \xi^\mu), \qquad \Delta_{\text{ideal}} = 0, \end{aligned} \tag{A4}$$

and the "first law of thermodynamics,"

$$\mathrm{d}E = T\mathrm{d}S + \mu \mathrm{d}Q - R_s \mathrm{d}\mu_s, \tag{A5}$$

giving physical meaning to the quantities we have introduced in Eq. (A1). Finally, we have the full set of superfluid constitutive relations up to ideal order satisfying the second law,

$$\begin{aligned} T^{\mu\nu} &= (E+P)u^\mu u^\nu + P g^{\mu\nu} + R_s \xi^\mu \xi^\nu + \mathcal{O}(\partial), \\ J^\mu &= Q u^\mu - R_s \xi^\mu + \mathcal{O}(\partial), \\ K &= -\alpha\delta_{\mathcal{B}}\varphi + \nabla_\mu (R_s \xi^\mu) + \mathcal{O}(\partial), \\ N^\mu &= \frac{P}{T} u^\mu + \delta_{\mathcal{B}}\varphi R_s \xi^\mu + \mathcal{O}(\partial), \\ J^\mu_S &= N^\mu - \frac{1}{T}(T^{\mu\nu}u_\nu + \mu J^\mu) = S u^\mu + \mathcal{O}(\partial), \\ \Delta &= \mathcal{O}(\partial^2). \end{aligned} \tag{A6}$$

These are the well known ideal superfluid constitutive relations. Note that we have included first-order terms in K, N^μ which can be ignored when talking about the ideal order, but are required for internal consistency with Eq. (2.16). The φ equation of motion $K = 0$ will imply

$$\begin{aligned} \alpha\delta_{\mathcal{B}}\varphi = \nabla_\mu (R_s \xi^\mu) + \mathcal{O}(\partial) \Rightarrow u^\mu \xi_\mu \\ = \mu + \frac{T}{\alpha} \nabla_\mu (R_s \xi^\mu) + \mathcal{O}(\partial), \end{aligned} \tag{A7}$$

which is a first-order correction to the Josephson equation. Note, however, that this equation can admit further one derivative corrections due to the first-order constitutive relations discussed in the next subsection; the correction mentioned here is only how the ideal superfluid transport affects the Josephson equation. The conservation laws on the other hand are complete up to the first order in derivatives,

$$\begin{aligned} &\frac{1}{\sqrt{-g}} \delta_{\mathcal{B}} (\sqrt{-g}(E+P)T^2 \beta_\mu) + QT\delta_{\mathcal{B}} A_\mu \\ &\quad = -\xi_\nu \alpha \delta_{\mathcal{B}}\varphi + \mathcal{O}(\partial^2), \\ &\frac{1}{\sqrt{-g}} \delta_{\mathcal{B}} (\sqrt{-g} QT) = \alpha\delta_{\mathcal{B}}\varphi + \mathcal{O}(\partial^2). \end{aligned} \tag{A8}$$

These equations provide a set of relations between $\delta_{\mathcal{B}}\varphi$, $\delta_{\mathcal{B}} g_{\mu\nu}$ and $\delta_{\mathcal{B}} A_\mu$, which can be used to eliminate a vector $u^\mu \delta_{\mathcal{B}} g_{\mu\nu}$ and a scalar $u^\mu \delta_{\mathcal{B}} A_\mu$ (see Table II) from the first-order constitutive relations. On the other hand, we choose to eliminate the scalar data $\nabla_\mu(R_s\xi^\mu)$ using the φ equation of motion.

2. First-order corrections to relativistic superfluids

Moving on to the one derivative superfluids, let us schematically represent various quantities appearing in Eq. (2.16) up to the first order in derivatives as

$$
\begin{aligned}
T^{\mu\nu} &= [(E+P)u^\mu u^\nu + Pg^{\mu\nu} + R_s\xi^\mu\xi^\nu] + \mathcal{T}^{\mu\nu} + \mathcal{O}(\partial^2),\\
J^\mu &= [Qu^\mu - R_s\xi^\mu] + \mathcal{J}^\mu + \mathcal{O}(\partial^2),\\
K &= [-\alpha\delta_\mathcal{B}\varphi + \nabla_\mu(R_s\xi^\mu)] + \mathcal{K} + \mathcal{O}(\partial^2),\\
N^\mu &= \left[\frac{P}{T}u^\mu + \delta_\mathcal{B}\varphi R_s\xi^\mu\right] + \mathcal{N}^\mu + \mathcal{O}(\partial^2),\\
\Delta &= \alpha(\delta_\mathcal{B}\varphi)^2 + \mathcal{D},
\end{aligned}
\tag{A9}
$$

where the corrections $\mathcal{T}^{\mu\nu}$, $\mathcal{J}^\mu$, $\mathcal{K}$, $\mathcal{N}^\mu$, $\mathcal{D}$ have exactly one derivative in every term. Plugging these in the Eq. (2.16) we can get an equation among the corrections,

$$
\begin{aligned}
\nabla_\mu\mathcal{N}^\mu - \mathrm{N}_\mathrm{H}^\perp = \frac{1}{2}\mathcal{T}^{\mu\nu}\delta_\mathcal{B}g_{\mu\nu} + \mathcal{J}^\mu\delta_\mathcal{B}A_\mu + \mathcal{K}\delta_\mathcal{B}\varphi\\
+ \mathcal{D} + \mathcal{O}(\partial^3).
\end{aligned}
\tag{A10}
$$

We will now attempt to find all the solutions to this equation, hence recovering the superfluid constitutive relations up to the first order in derivatives.

a. Parity-even

We can find the most general parity-even solution of Eq. (A10) in two steps (note that $\mathrm{N}_\mathrm{H}^\perp$ is parity odd): (1) first, we write down the most general allowed parity-even $\mathcal{N}^\mu$ and find a set of constitutive relations pertaining to that, and (2) we find the most general parity-even constitutive relations which satisfy Eq. (A10) with $\mathcal{N}^\mu = 0$.

(1) One can check that the most general form of $\mathcal{N}^\mu$ (whose divergence only contains product of derivatives and has at least one $\delta_\mathcal{B}$ per term) can be written as

$$
\begin{aligned}
\mathcal{N}^\mu = 2f_1 u^{[\mu}\xi^{\nu]}\frac{1}{T^2}\partial_\nu T + 2f_2 u^{[\mu}\xi^{\nu]}\partial_\nu\left(\frac{\mu}{T}\right)\\
+ 2f_3 u^{[\mu}\xi^{\nu]}\partial_\nu R_s + \nabla_\nu(f_4 u^{[\mu}\xi^{\nu]}),
\end{aligned}
\tag{A11}
$$

where f's are functions of T, $\nu = \mu/T$ and $\hat{\mu}_s = -\frac{1}{2}\zeta^\mu\zeta_\mu$ with $\zeta^\mu = P^{\mu\nu}\xi_\nu$ ($P^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$ is the projection operator away from the fluid velocity). Note that

$$
\begin{aligned}
\hat{\mu}_s &= -\frac{1}{2}\zeta^\mu\zeta_\mu = -\frac{1}{2}\xi^\mu\xi_\mu - \frac{1}{2}(\xi^\mu u_\mu)^2\\
&= \mu_s - \frac{1}{2}(\mu + T\delta_\mathcal{B}\varphi)^2.
\end{aligned}
\tag{A12}
$$

Out of the four terms in Eq. (A11), the last one has trivially zero divergence and hence can be ignored. The third term on the other hand can be removed by elimination of $\nabla_\mu(R_s\xi^\mu)$ using the φ equation of motion. Computing the divergence of the remaining terms in $\mathcal{N}^\mu$ and comparing them to Eq. (A10), we can directly read out the corresponding superfluid constitutive relations (the symbol '$\ni$' represents that they are not yet the complete solutions of Eq. (A10); we still have to add the terms with $\mathcal{N}^\mu = 0$),

$$
\begin{aligned}
\mathcal{T}^{\mu\nu} &\ni u^\mu u^\nu\left(\sum_{i=1}^{2}\alpha_{E,i}S_{e,i} - \frac{1}{T}\nabla_\sigma(Tf_1\zeta^\sigma)\right) + (\zeta^\mu\zeta^\nu - 2(u^\rho\xi_\rho)u^{(\mu}\zeta^{\nu)})\sum_{i=1}^{2}\alpha_{R_s,i}S_{e,i}\\
&\quad + \tilde{P}^{\mu\nu}\sum_{i=1}^{2}f_i S_{e,i} - 2\xi^{(\mu}\sum_{i=1}^{2}f_i V_{e,1}^{\nu)} + 2u^{(\mu}\zeta^{\nu)}\sum_{i=1}^{2}f_i S_{4+i},\\
\mathcal{J}^\mu &\ni u^\mu\left(\sum_{i=1}^{2}\alpha_{Q,i}S_{e,i} - \frac{1}{T}\nabla_\nu(Tf_2\zeta^\nu)\right) - \zeta^\mu\sum_{i=1}^{2}\alpha_{R_s,i}S_{e,i} + \sum_{i=1}^{2}f_i V_{e,i}^\mu,\\
\mathcal{K} &\ni \nabla_\mu\left(\zeta^\mu\sum_{i=1}^{2}\alpha_{R_s,i}S_{e,i} - \sum_{i=1}^{2}f_i V_{e,i}^\mu\right),
\end{aligned}
\tag{A13}
$$

where $\tilde{P}^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu - \frac{1}{\zeta^\sigma\zeta_\sigma}\zeta^\mu\zeta^\nu$, and we have defined

$$
\mathrm{d}f_i = \frac{\alpha_{E,i}}{T}\mathrm{d}T + T\alpha_{Q,i}\mathrm{d}\nu + \left(\alpha_{R_s,i} - \frac{f_i}{2\hat{\mu}_s}\right)\mathrm{d}\hat{\mu}_s.
\tag{A14}
$$

The actual computation is not neat and we have presented the details in Appendix C for the aid of the readers interested in reproducing our results. Note that these constitutive relations are presented in terms of "data" that are natural for this sector; readers can modify these to their favorite basis and get results which might look considerably messier. Moreover, these results are written in a particular 'hydrodynamic frame' chosen by aligning u^μ, T, μ along β^μ, Λ_β, which again can be modified according to reader's preference.

(2) Let us now look at the parity-even solutions to Eq. (A10) with $\mathcal{N}^\mu = 0$,

$$0 = \frac{1}{2}\mathcal{T}^{\mu\nu}\delta_{\mathcal{B}}g_{\mu\nu} + \mathcal{J}^\mu\delta_{\mathcal{B}}A_\mu + \mathcal{K}\delta_{\mathcal{B}}\varphi + \mathcal{D}. \tag{A15}$$

Every term in $\mathcal{T}^{\mu\nu}$, $\mathcal{J}^\mu$, $\mathcal{K}$ must either cancel or contribute to Δ which has to be a quadratic form. It follows that the terms in $\mathcal{T}^{\mu\nu}$, $\mathcal{J}^\mu$, $\mathcal{K}$ must be proportional to $\delta_{\mathcal{B}}g_{\mu\nu}$, $\delta_{\mathcal{B}}A_\mu$, $\delta_{\mathcal{B}}\varphi$. Recall, however, that we have chosen to eliminate $u^\mu\delta_{\mathcal{B}}g_{\mu\nu}$, $u^\mu\delta_{\mathcal{B}}A_\mu$ using the equations of motion. For Δ to be a quadratic form, it therefore implies that $\mathcal{T}^{\mu\nu}$, $\mathcal{J}^\mu$ cannot have a term like $\#^{(\mu}u^{\nu)}$, $\#u^\mu$ respectively for some vector $\#^\mu$ and scalar $\#$. With this input let us write down the most generic allowed form of the currents in terms of 20 new transport coefficients $[\beta_{ij}]_{4\times4}$ (with $\beta_{44} = \alpha/T$), $[\kappa_{ij}]_{2\times2}$ and η,

$$\begin{aligned}\mathcal{T}^{\mu\nu} \ni& -T[\{\beta_{11}\tilde{P}^{\rho\sigma} + \beta_{12}\zeta^\rho\zeta^\sigma\}\tilde{P}^{\mu\nu} + \{\beta_{21}\tilde{P}^{\mu\nu} + \beta_{22}\zeta^\rho\zeta^\sigma\}\zeta^\mu\zeta^\nu + 4\kappa_{11}\zeta^{(\mu}\tilde{P}^{\nu)(\rho}\zeta^{\sigma)} \\ &+ \eta\tilde{P}^{\mu\langle\rho}\tilde{P}^{\sigma\rangle\nu}]\frac{1}{2}\delta_{\mathcal{B}}g_{\rho\sigma} - T[\beta_{13}\zeta^\rho\tilde{P}^{\mu\nu} + \beta_{23}\zeta^\rho\zeta^\mu\zeta^\nu + 2\kappa_{12}\zeta^{(\mu}\tilde{P}^{\nu)\rho}]\delta_{\mathcal{B}}A_\rho \\ &- T[\beta_{14}\tilde{P}^{\mu\nu} + \beta_{24}\zeta^\mu\zeta^\nu]\delta_{\mathcal{B}}\varphi, \\ =& -\tilde{P}^{\mu\nu}\sum_{i=1}^4\beta_{1i}S_i - \zeta^\mu\zeta^\nu\sum_{i=1}^4\beta_{2i}S_i - 2\zeta^{(\mu}\sum_{i=1}^2\kappa_{1i}V_i^{\nu)} - \eta\sigma^{\mu\nu}.\end{aligned} \tag{A16}$$

$$\begin{aligned}\mathcal{J}^\mu \ni& -T[\{\beta_{31}\tilde{P}^{\rho\sigma} + \beta_{32}\zeta^\rho\zeta^\sigma\}\zeta^\mu + 2\kappa_{21}\tilde{P}^{\mu(\rho}\zeta^{\sigma)}]\frac{1}{2}\delta_{\mathcal{B}}g_{\rho\sigma} \\ &- T[\beta_{33}\zeta^\rho\zeta^\mu + \kappa_{22}\tilde{P}^{\mu\rho}]\delta_{\mathcal{B}}A_\rho - T[\beta_{34}\zeta^\mu]\delta_{\mathcal{B}}\varphi, \\ =& -\zeta^\mu\sum_{i=1}^4\beta_{3i}S_i - \sum_{i=1}^2\kappa_{2i}V_i^\mu,\end{aligned} \tag{A17}$$

$$\mathcal{K} \ni -T[\beta_{41}\tilde{P}^{\rho\sigma} + \beta_{42}\zeta^\rho\zeta^\sigma]\delta_{\mathcal{B}}g_{\rho\sigma} - T[\beta_{43}\zeta^\rho]\delta_{\mathcal{B}}A_\rho = -\sum_{i=1}^3\beta_{4i}S_i. \tag{A18}$$

Note that we did not include a term proportional to $\delta_{\mathcal{B}}\varphi$ in $\mathcal{K}$, because such a term is already present in $K = -\alpha\delta_{\mathcal{B}}\varphi + \nabla_\mu(R_s\xi^\mu) + \mathcal{K} + \mathcal{O}(\partial^2)$. Defining $\beta_{44} = \alpha/T$, we can read out the parity-even quadratic form $\Delta|_{\text{even}} = \alpha(\delta_{\mathcal{B}}\varphi)^2 + \mathcal{D}|_{\text{even}}$,

$$\begin{aligned}T\Delta|_{\text{even}} &= \sum_{i,j=1}^4 S_i\beta_{ij}S_j + \sum_{i,j=1}^2 V_i^\mu\kappa_{ij}V_{i,\mu} + \eta\sigma^{\mu\nu}\sigma_{\mu\nu}, \\ &= \sum_{i,j=1}^4 S_i\beta_{(ij)}S_j + \sum_{i,j=1}^2 V_i^\mu\kappa_{(ij)}V_{i,\mu} + \eta\sigma^{\mu\nu}\sigma_{\mu\nu}.\end{aligned} \tag{A19}$$

In the second step we have realized that only the symmetric parts of the matrices β_{ij} and κ_{ij} will survive in this expression, and will contribute towards dissipation. Thus only 14 out of 21 transport coefficients (including α) are dissipative; the remaining 7 are nondissipative.

b. Parity-odd (four dimensions)

We can find the most general parity-odd solution of Eq. (A10) in three steps: (1) first, we consider a particular set of solutions which takes care of the anomaly $\mathrm{N}_{\mathrm{H}}^{\perp}$ and proceed towards the nonanomalous constitutive relations, (2) then we write down the most general allowed parity-odd $\mathcal{N}^\mu$ and find a set of constitutive relations pertaining to that, and (3) we find the most general parity-odd constitutive relations with zero $\mathcal{N}^\mu$.

(1) In four dimensions at the first order in the derivatives $\mathrm{T}_{\mathrm{H}}^{\mu\perp} = 0$ and $\mathrm{J}_{\mathrm{H}}^{\perp} = -\frac{3}{4}C^{(4)}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$, which implies,

$$\mathrm{N}_{\mathrm{H}}^{\perp} = -\frac{3}{4}\nu C^{(4)}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}. \tag{A20}$$

A particular solution pertaining to Eq. (A10) with this $\mathrm{N}_{\mathrm{H}}^{\perp}$ is given as (see e.g. [26]),

$$\begin{aligned}
\mathcal{T}^{\mu\nu} &\ni 2\mu^2 C^{(4)} u^{(\mu}(3M^{\nu)} + 2\mu\omega^{\nu)}),\\
\mathcal{J}^{\mu} &\ni \mu C^{(4)}(6M^{\mu} + 3\mu\omega^{\mu}),\\
\mathcal{K} &\ni 0,\\
\mathcal{N}^{\mu} &\ni \frac{\mu^2}{T}C^{(4)}(3M^{\mu} + \mu\omega^{\mu}).
\end{aligned} \tag{A21}$$

Here we have defined the magnetic field and fluid vorticity as

$$M^{\mu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}u_{\nu}F_{\rho\sigma}, \qquad \omega^{\mu} = \epsilon^{\mu\nu\rho\sigma}u_{\nu}\partial_{\rho}u_{\sigma}. \tag{A22}$$

(2) One can check that the most general form of $\mathcal{N}^{\mu}$ (whose divergence only contains product of derivatives and has at least one $\delta_{\mathcal{B}}$ per term) can be written as

$$\begin{aligned}
\mathcal{N}^{\mu} = {}& g_1(\beta^{\mu}\tilde{S}_{e,1} + \tilde{V}_3^{\mu}) + g_2(\beta^{\mu}\tilde{S}_{e,2} + \tilde{V}_2^{\mu})\\
&+ C_1 T^2\omega^{\mu},
\end{aligned} \tag{A23}$$

where g's are functions of T, ν, $\hat{\mu}_s$, and C_1 is a constant. From here we can directly read out the corresponding constitutive relations,

$$\begin{aligned}
\mathcal{T}^{\mu\nu} \ni{}& u^{\mu}u^{\nu}\sum_{i=1}^{2}\tilde{\alpha}_{E,i}\tilde{S}_{e,i} + (\zeta^{\mu}\zeta^{\nu} - 2(u^{\rho}\xi_{\rho})u^{(\mu}\zeta^{\nu)})\sum_{i=1}^{2}\tilde{\alpha}_{R_s,i}\tilde{S}_{e,i} - \zeta^{\mu}\zeta^{\nu}\sum_{i=1}^{2}\frac{1}{2\hat{\mu}_s}g_i\tilde{S}_{e,i}\\
&- 2u^{(\mu}\sum_{i=1}^{2}g_i\tilde{V}_{e,2+i}^{\nu)} - u^{(\mu}(2P_{\alpha}^{\nu)} - u^{\nu)}u_{\alpha})\epsilon^{\alpha\rho\sigma\tau}\nabla_{\sigma}(Tg_1u_{\tau}\zeta_{\rho}) + 4C_1T^3\omega^{(\mu}u^{\nu)}\\
\mathcal{J}^{\mu} \ni{}& u^{\mu}\sum_{i=1}^{2}\tilde{\alpha}_{Q,i}\tilde{S}_{e,i} - \zeta^{\mu}\sum_{i=1}^{2}\tilde{\alpha}_{R_s,i}\tilde{S}_{e,i} + \sum_{i=1}^{2}g_i\tilde{V}_{e,i}^{\mu} + \epsilon^{\mu\nu\rho\sigma}\nabla_{\nu}(Tg_2\zeta_{\rho}u_{\sigma}),\\
\mathcal{K} \ni{}& \nabla_{\mu}\left(\zeta^{\mu}\sum_{i=1}^{2}\tilde{\alpha}_{R_s,i}\tilde{S}_{e,i} - \sum_{i=1}^{2}g_i\tilde{V}_{e,i}^{\mu}\right),
\end{aligned} \tag{A24}$$

where we have defined,

$$\mathrm{d}g_i = \frac{\tilde{\alpha}_{E,i}}{T}\mathrm{d}T + T\tilde{\alpha}_{Q,i}\mathrm{d}\nu + \left(\tilde{\alpha}_{R_s,i} - \frac{g_i}{2\hat{\mu}_s}\right)\mathrm{d}\hat{\mu}_s. \tag{A25}$$

The actual computation is not neat and we have presented the details in Appendix C for interested readers.

(3) We should finally consider the parity-odd constitutive relations that satisfy Eq. (A10) with zero lhs. Following our discussion in the parity-even sector, the allowed form of the constitutive relations can be written down in terms of five coefficients $[\tilde{\kappa}_{ij}]_{2\times 2}$ and $\tilde{\eta}$,

$$\begin{aligned}
\mathcal{T}^{\mu\nu} &\ni -Tu_{\tau}\zeta_{\kappa}[4\tilde{\kappa}_{11}\zeta^{(\mu}\epsilon^{\nu)\tau\kappa(\rho}\zeta^{\sigma)} + \tilde{\eta}\tilde{P}^{\lambda(\mu}\epsilon^{\nu)\tau\kappa(\rho}\tilde{P}_{\lambda}^{\sigma)}]\frac{1}{2}\delta_{\mathcal{B}}g_{\rho\sigma} - Tu_{\tau}\zeta_{\kappa}[2\tilde{\kappa}_{12}\zeta^{(\mu}\epsilon^{\nu)\tau\kappa\rho}]\delta_{\mathcal{B}}A_{\rho},\\
&= -2\zeta^{(\mu}\sum_{i=1}^{2}\tilde{\kappa}_{1i}\tilde{V}_i^{\nu)} - \tilde{\eta}\tilde{\sigma}^{\mu\nu},\\
\mathcal{J}^{\mu} &\ni -Tu_{\tau}\zeta_{\kappa}[2\tilde{\kappa}_{21}\epsilon^{\mu\tau\kappa(\rho}\zeta^{\sigma)}]\frac{1}{2}\delta_{\mathcal{B}}g_{\rho\sigma} - Tu_{\tau}\zeta_{\kappa}[\tilde{\kappa}_{22}\epsilon^{\mu\tau\kappa\rho}]\delta_{\mathcal{B}}A_{\rho},\\
&\ni -\sum_{i=1}^{2}\tilde{\kappa}_{2i}\tilde{V}_i^{\mu},\\
\mathcal{K} &\ni 0.
\end{aligned} \tag{A26}$$

One can check that these constitutive relations trivially satisfy Eq. (A10) with zero lhs and the quadratic form $\Delta|_{\mathrm{odd}} = \mathcal{D}|_{\mathrm{odd}}$ is given as

$$
\begin{aligned}
T\Delta|_{\text{odd}} &= \epsilon^{\mu\nu\tau\kappa} u_\tau \zeta_\kappa \left[\sum_{i=1}^{2} V_{i,\mu} \tilde{\kappa}_{ij} V_{j,\nu} + \tilde{\eta} \sigma_{\mu\rho} \sigma_\nu^\rho \right] \\
&= \epsilon^{\mu\nu\tau\kappa} u_\tau \zeta_\kappa \sum_{i=1}^{2} V_{i,\mu} \tilde{\kappa}_{[ij]} V_{j,\nu} \\
&= 2\epsilon^{\mu\nu\tau\kappa} u_\tau \zeta_\kappa V_{1,\mu} \tilde{\kappa}_{[12]} V_{2,\nu}.
\end{aligned} \tag{A27}
$$

It follows that out of the 5 transport coefficients, only 1 contribute to dissipation and the other 4 are nondissipative.

c. Positivity constraints

The dissipative transport coefficients are required to satisfy a set of inequalities to satisfy $\Delta = \alpha(\delta_{\mathcal{B}}\varphi)^2 + \mathcal{D}|_{\text{even}} + \mathcal{D}|_{\text{odd}} \geq 0$,

$$
\begin{aligned}
T\Delta = \sum_{i,j=1}^{4} S_i \beta_{(ij)} S_j + \left(\sum_{i,j=1}^{2} V_i^\mu \kappa_{(ij)} V_{i,\mu} + \sum_{i=1}^{2} V_i^\mu \tilde{\kappa}_{[ij]} \tilde{V}_{j,\mu} \right) \\
+ \eta \sigma^{\mu\nu} \sigma_{\mu\nu}.
\end{aligned} \tag{A28}
$$

We want this expression to be a quadratic form, which it nearly is except the parity-odd term in the brackets. However this term can be made into a quadratic form by noticing that the square of a parity-odd term is parity-even, due to the identity,

$$
(\epsilon^{\mu\nu\rho\sigma} u_\rho \zeta_\sigma)(\epsilon_{\tau\nu\alpha\beta} u^\alpha \zeta^\beta) = \tilde{P}^\mu_\tau \zeta^\nu \zeta_\nu = -2\hat{\mu}_s \tilde{P}^\mu_\tau. \tag{A29}
$$

We define,

$$
\begin{aligned}
&\begin{pmatrix} V'^\mu_1 \\ V'^\mu_2 \end{pmatrix} = \begin{pmatrix} V^\mu_1 \\ V^\mu_2 \end{pmatrix} + \begin{pmatrix} 0 & a_{12} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \tilde{V}^\mu_1 \\ \tilde{V}^\mu_2 \end{pmatrix}, \\
&\kappa'_{ij} = \kappa_{ij} + k_{ij}, \qquad k_{[ij]} = 0,
\end{aligned} \tag{A30}
$$

such that,

$$
\sum_{i,j=1}^{2} V'^\mu_i \kappa'_{(ij)} V'_{i,\mu} = \sum_{i,j=1}^{2} V^\mu_i \kappa_{(ij)} V_{i,\mu} + \sum_{i=1}^{2} V^\mu_i \tilde{\kappa}_{[ij]} \tilde{V}_{j,\mu}. \tag{A31}
$$

Using the identity Eq. (A29), the above equation can be easily solved to give,

$$
a_{12} = \frac{\tilde{\kappa}^{(a)}_{12}}{\kappa_{11}}, \qquad k_{1i} = k_{i1} = 0, \qquad k_{22} = 2\hat{\mu}_s \frac{\tilde{\kappa}_{[12]}}{\kappa_{11}}. \tag{A32}
$$

Consequently Δ will take the form,

$$
T\Delta = \sum_{i,j=1}^{4} S_i \beta_{(ij)} S_j + \sum_{i,j=1}^{2} V'^\mu_i \kappa'_{(ij)} V'_{i,\mu} + \eta \sigma^{\mu\nu} \sigma_{\mu\nu}. \tag{A33}
$$

Given $T \geq 0$, the condition $\Delta \geq 0$ implies that $\eta \geq 0$ and the matrices $[\beta_{(ij)}]_{4\times 4}$, $[\kappa'_{(ij)}]_{2\times 2}$ have all non-negative eigenvalues. This gives 7 inequalities among 15 dissipative transport coefficients, and 8 are completely arbitrary.

This finishes the off-shell formalism derivation of the constitutive relations of a relativistic superfluid up to first order in derivatives. A concise summary of these results has been presented in Sec. II C.

APPENDIX B: EQUILIBRIUM PARTITION FUNCTION FOR NULL SUPERFLUIDS

It was realized by [29,30] that a huge part of the (super) fluid constitutive relations can be fixed by requiring existence of an equilibrium partition function, which generates the part of the constitutive relations that survive in equilibrium. In this Appendix, we will discuss the equilibrium partition function for Galilean superfluids. In hydrodynamics, equilibrium is defined by a set of fields $\mathcal{K} = \{K^M, \Lambda_K\}$ with $K^M K_M < 0$, which act on the background fields g_{MN}, A_M and the superfluid phase φ as an isometry,

$$
\begin{aligned}
\delta_{\mathcal{K}} g_{MN} &= \nabla_M K_N + \nabla_N K_M = 0, \\
\delta_{\mathcal{K}} A_M &= \partial_M(\Lambda_K + K^N A_N) + K^N F_{NM} = 0, \\
\delta_{\mathcal{K}} \varphi &= K^M \partial_M \varphi - \Lambda_K = K^M \xi_M - (\Lambda_K + K^N A_N) = 0.
\end{aligned} \tag{B1}
$$

For simplicity, we choose a basis $\{x^M\} = \{x^-, t, x^i\}$ such that the null isometry $\mathcal{V} = \{V = \partial_-, \Lambda_V = 0\}$ and the equilibrium isometry $\mathcal{K} = \{K = \partial_t, \Lambda_K = 0\}$. The fact that $\mathcal{V}, \mathcal{K}$ are isometries implies that all the fields are independent of x^-, t coordinates. In this basis, we decompose the background fields as

$$
\begin{aligned}
\mathrm{d}s^2 &= -2\mathrm{e}^{-\Phi}(\mathrm{d}t + a_i \mathrm{d}x^i)(\mathrm{d}x^- - B_t \mathrm{d}t - B_i \mathrm{d}x^i) \\
&\quad + g_{ij} \mathrm{d}x^i \mathrm{d}x^j, \\
A &= -\mathrm{d}x^- + A_t \mathrm{d}t + A_i \mathrm{d}x^i.
\end{aligned} \tag{B2}
$$

We will denote the covariant derivative associated with the spatial metric g_{ij} by $\acute{\nabla}_i$. After choosing the said basis, the residual diffeomorphisms are the spatial diffeomorphisms $x^i \to x^i + \chi^i(x^j)$, mass gauge transformations $x^- \to x^- + \chi^-(x^i)$ and Kaluza-Klein gauge transformations $t \to t + \chi^t(x^i)$. Under mass gauge transformations, only fields that transform are,

$$
\delta_{\chi^-} B_i = -\partial_i \chi^-, \qquad \delta_{\chi^-} A_i = -\partial_i \chi^-, \tag{B3}
$$

while under Kaluza-Klein gauge transformations,

$$
\delta_{\chi^+} a_i = \partial_i \chi^+, \qquad \delta_{\chi^+} B_i = B_t \partial_i \chi^+, \qquad \delta_{\chi^+} A_i = A_t \partial_i \chi^+. \tag{B4}
$$

We define the fields,

$$\acute{B}_i = B_i - a_i B_t, \qquad \acute{A}_i = A_i - a_i A_t - \acute{B}_i. \tag{B5}$$

$\acute{B}_i$ is mass gauge field which is invariant under Kaluza-Klein guage transformations. $\acute{A}_i$ on the other hand is invariant under both mass and Kaluza-Klein gauge transformations, and only transforms under the U(1). a_i is Kaluza-Klein gauge field. Components of the superfluid velocity $\xi_M = \partial_M\varphi + A_M$ can be found as

$$\xi_- = -1, \qquad \xi_t = A_t, \qquad \xi_i = \partial_i\varphi + A_i. \tag{B6}$$

Out of these, ξ_i is not mass or Kaluza-Klein gauge invariant due to presence of A_i. We can write an invariant version as

$$\acute{\xi}_i = \partial_i\varphi + \acute{A}_i. \tag{B7}$$

The superfluid potential can also be written in terms of these as

$$\mu_s = -\frac{1}{2}\xi^M\xi_M = -\frac{1}{2}\acute{\xi}^i\acute{\xi}_i - e^{\Phi}A_t + e^{\Phi}B_t, \tag{B8}$$

and we define $\acute{\mu}_s = -\frac{1}{2}\acute{\xi}^i\acute{\xi}_i$. Finally, the fundamental variables at equilibrium are,

$$\Phi, \quad A_t, \quad B_t, \quad a_i, \quad \acute{A}_i, \quad \acute{B}_i, \quad g_{ij}, \quad \varphi. \tag{B9}$$

The argument is that at equilibrium, constitutive relations should be derivable from an equilibrium partition function written in terms of these fundamental fields. In covariant terms, variation of an equilibrium partition function W can be parametrized as

$$\delta W = \int \{dx^M\}\sqrt{-g}\left(\frac{1}{2}T^{MN}\delta g_{MN} + J^M\delta A_M + K\delta\varphi\right). \tag{B10}$$

In our chosen basis it decomposes as

$$\begin{aligned}\delta W = \int \{dx^i\}\sqrt{g_3}\Big[&(T_{t-} + T_{--}B_t)\delta\Phi + e^{-\Phi}(T^i{}_t + J^iA_t)\delta a_i + \frac{1}{2}e^{-\Phi}T^{ij}\delta g_{ij}\\ &+ (T_{--}\delta B_t - e^{-\Phi}(T^i{}_- - J^i)\delta\acute{B}_i) - (J_-\delta A_t - e^{-\Phi}J^i\delta\acute{A}_i) + e^{-\Phi}K\delta\varphi\Big],\end{aligned} \tag{B11}$$

where $g_3 = \det g_{ij}$. Now, given the most generic partition function $W[\Phi, A_t, B_t, a_i, \acute{A}_i, \acute{B}_i, g_{ij}, \varphi]$ as a gauge invariant scalar functional of the fundamental fields, various components of the currents T^{MN}, J^M, K can be read out in terms of W as

$$\begin{aligned}&T_{--} = \frac{1}{\sqrt{g_3}}\frac{\delta W}{\delta B_t}, \qquad T_{t-} = \frac{1}{\sqrt{g_3}}\left(\frac{\delta W}{\delta\Phi} - B_t\frac{\delta W}{\delta B_t}\right),\\ &T^i_- = -\frac{e^{\Phi}}{\sqrt{g_3}}\left(\frac{\delta W}{\delta\acute{B}_i} - \frac{\delta W}{\delta\acute{A}_i}\right), \qquad T^i_t = \frac{e^{\Phi}}{\sqrt{g_3}}\left(\frac{\delta W}{\delta a_i} - A_t\frac{\delta W}{\delta\acute{A}_i}\right), \qquad T^{ij} = \frac{2e^{\Phi}}{\sqrt{g_3}}\frac{\delta W}{\delta g_{ij}},\\ &J_- = -\frac{1}{\sqrt{g_3}}\frac{\delta W}{\delta A_t}, \qquad J^i = \frac{e^{\Phi}}{\sqrt{g_3}}\frac{\delta W}{\delta\acute{A}_i}.\end{aligned} \tag{B12}$$

Since these expressions are already in a "noncovariant notation", we can easily perform null reduction to read out the Galilean currents. We define a Galilean frame field to perform the reduction,

$$v^M_{(K)} = -\frac{K^M}{V_MK^M} + \frac{K^RK_RV^M}{2(V_NK^N)^2} = \begin{pmatrix} e^{\Phi}B_t \\ e^{\Phi} \\ 0 \end{pmatrix}. \tag{B13}$$

In $v^M_{(K)}$ Galilean frame, the Galilean currents can be read out in terms of W as

$$\begin{aligned}&\rho = \frac{1}{\sqrt{g_3}}\frac{\delta W}{\delta B_t}, \qquad \rho^i = \frac{e^{\Phi}}{\sqrt{g_3}}\left(\frac{\delta W}{\delta\acute{B}_i} - \frac{\delta W}{\delta\acute{A}_i}\right), \qquad t^{ij}_{(v_K)} = \frac{2e^{\Phi}}{\sqrt{g_3}}\frac{\delta W}{\delta g_{ij}},\\ &\epsilon_{(v_K)} = \frac{e^{\Phi}}{\sqrt{g_3}}\frac{\delta W}{\delta\Phi}, \qquad \epsilon^i_{(v_K)} = \frac{e^{2\Phi}}{\sqrt{g_3}}\left(-\frac{\delta W}{\delta a_i} + (A_t - B_t)\frac{\delta W}{\delta\acute{A}_i} + B_t\frac{\delta W}{\delta\acute{B}_i}\right),\\ &j = \frac{1}{\sqrt{g_3}}\frac{\delta W}{\delta A_t}, \qquad j^i = \frac{e^{\Phi}}{\sqrt{g_3}}\frac{\delta W}{\delta\acute{A}_i}.\end{aligned} \tag{B14}$$

Finally, we can write down the most general equilibrium partition function W up to first order in derivatives as

$$W = \int \{\mathrm{d}x^i\}\sqrt{g_3}\Big[\mathrm{e}^{-\Phi}P + \mathrm{e}^{-\Phi}f_1\acute{\xi}^i\partial_i\Phi + f_2\acute{\xi}^i\partial_iA_t + f_3\acute{\xi}^i\partial_iB_t + f_4\acute{\nabla}_i\left(\acute{\xi}^i\frac{\partial P}{\partial\acute{\mu}_s}\right) + \acute{\nabla}_i(f_5\acute{\xi}^i) \\ + (g_1+g_2)\varepsilon^{ijk}\acute{\xi}_i\partial_j\acute{B}_k + g_2\varepsilon^{ijk}\acute{\xi}_i\partial_j\acute{A}_k + (g_1B_t + g_2A_t - \mathrm{e}^{-\Phi}g_3)\varepsilon^{ijk}\acute{\xi}_i\partial_ja_k - C_1\varepsilon^{ijk}a_i\partial_j\acute{B}_k\Big], \tag{B15}$$

where the coefficients P, f_i, g_i are arbitrary functions of the scalars Φ, A_t, B_t and $\acute{\mu}_s$. C_1 on the other hand has to be a constant, so that integral of the term coupling to it is gauge invariant. The term coupling to f_4 is multiplied with the first-order equation of motion of φ and hence can be neglected. On the other hand, term coupling to f_5 is a total derivative. Acute reader might note that we have not included a term like to $C_0\varepsilon^{ijk}\acute{B}_i\partial_j\acute{B}_k$. The reason is that this term does not have a "covariant analogue" and hence is switched off by the second law of thermodynamics [17]. Finally, this equilibrium partition function does not account for anomalies; for a discussion on anomalous partition function for null fluids see [17,22].

Varying the partition function W in Eq. (B15) and using Eq. (B14), we can read out the equilibrium constitutive relations. We will not perform the explicit variation here, but one can check that the constitutive relations gained are the same as the ones derived in the bulk of the paper, after identifying the equilibrium values of the hydrodynamic fields,

$$u^M|_{\mathrm{eqb}} = v^M_{(K)}, \qquad T|_{\mathrm{eqb}} = \mathrm{e}^{\Phi}, \\ \mu_n|_{\mathrm{eqb}} = \mathrm{e}^{\Phi}B_t, \qquad \mu|_{\mathrm{eqb}} = \mathrm{e}^{\Phi}A_t. \tag{B16}$$

These can also be summarized as $\mathcal{B}|_{\mathrm{eqb}} = \{\beta^M, \Lambda_\beta\}_{\mathrm{eqb}} = \{K^M, \Lambda_K\} = \mathcal{K}$. Having established that, the equilibrium value of the projected superfluid velocity is given as

$$\zeta_M|_{\mathrm{eqb}} = P_{MN}\xi^N|_{\mathrm{eqb}} = \begin{pmatrix} 0 \\ 0 \\ \acute{\xi}_i \end{pmatrix}, \tag{B17}$$

and hence $\hat{\mu}_s|_{\mathrm{eqb}} = \acute{\mu}_s$. This finishes our discussion of equilibrium partition function for null/Galilean superfluids.

APPENDIX C: CALCULATIONAL DETAILS

In this Appendix, we will give details of the computation regarding divergence of the free energy current, glossed over in the main text. We will find the following identities useful in the following computation: let S be a scalar and β^μ be a vector, then,

$$\nabla_\mu(\beta^\mu S) = \frac{1}{\sqrt{-g}}\pounds_\beta(\sqrt{-g}S) = \frac{1}{2}Sg^{\mu\nu}\pounds_\beta g_{\mu\nu} + \pounds_\beta S. \tag{C1}$$

There is a corresponding null background version of this identity,

$$\nabla_M(\beta^M S) = \frac{1}{\sqrt{-g}}\pounds_\beta(\sqrt{-g}S) = \frac{1}{2}Sg^{MN}\pounds_\beta g_{MN} + \pounds_\beta S. \tag{C2}$$

Given a tensor $X^{\mu\nu}$, we have,

$$\nabla_\mu\nabla_\nu X^{[\mu\nu]} = \frac{1}{2}(\nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu)X^{\mu\nu} \\ = \frac{1}{2}(R_{\mu\nu}{}^{\mu}{}_{\rho}X^{\rho\nu} + R_{\mu\nu}{}^{\nu}{}_{\rho}X^{\mu\rho}) \\ = \frac{1}{2}(R_{\nu\rho}X^{\rho\nu} - R_{\mu\rho}X^{\mu\rho}) = 0. \tag{C3}$$

Similarly,

$$\nabla_M\nabla_N X^{[MN]} = 0. \tag{C4}$$

Relativistic superfluid free energy current: Let us start with relativistic superfluids. The $\delta_{\mathcal{B}}$ variation of hydrodynamic and superfluid fields can be computed to be,

$$\delta_{\mathcal{B}}T = \frac{T}{2}u^\mu u^\nu\delta_{\mathcal{B}}g_{\mu\nu}, \qquad \delta_{\mathcal{B}}\left(\frac{\mu}{T}\right) = \frac{1}{T}u^\mu\delta_{\mathcal{B}}A_\mu, \qquad \delta_{\mathcal{B}}\mu_s = \frac{1}{2}\xi^\mu\xi^\nu\delta_{\mathcal{B}}g_{\mu\nu} - \xi^\mu\delta_{\mathcal{B}}A_\mu - \xi^\mu\nabla_\mu\delta_{\mathcal{B}}\varphi, \\ \delta_{\mathcal{B}}\hat{\mu}_s = \frac{1}{2}(\zeta^\mu\zeta^\nu - 2(u^\rho\xi_\rho)u^{(\mu}\zeta^{\nu)})\delta_{\mathcal{B}}g_{\mu\nu} - \zeta^\mu\delta_{\mathcal{B}}A_\mu - \zeta^\mu\nabla_\mu\delta_{\mathcal{B}}\varphi, \\ \delta_{\mathcal{B}}u^\mu = \frac{1}{2}u^\mu u^\rho u^\nu\delta_{\mathcal{B}}g_{\rho\nu}, \qquad \delta_{\mathcal{B}}u_\mu = (2P^{(\rho}_\mu u^{\nu)} - u_\mu u^\rho u^\nu)\frac{1}{2}\delta_{\mathcal{B}}g_{\rho\nu}, \\ \delta_{\mathcal{B}}\zeta^\mu = (u^\mu u^{(\rho}\zeta^{\sigma)} - P^{\mu(\rho}\xi^{\sigma)})\delta_{\mathcal{B}}g_{\rho\sigma} + P^{\mu\nu}\delta_{\mathcal{B}}\xi_\nu, \qquad \delta_{\mathcal{B}}\zeta_\mu = (u^\nu\xi_\nu)u^{(\rho}P^{\sigma)}_\mu\delta_{\mathcal{B}}g_{\rho\sigma} + P^\nu_\mu\delta_{\mathcal{B}}\xi_\nu. \tag{C5}$$

The first-order parity-even free energy current $\mathcal{N}^\mu$ in Eq. (A11) has a term $2f_1 u^{[\mu}\xi^{\nu]}\frac{1}{T^2}\partial_\nu T$. We compute its divergence,

$$\begin{aligned}
\nabla_\mu\left(2f_1 u^{[\mu}\zeta^{\nu]}\frac{1}{T^2}\partial_\nu T\right) &= f_1\zeta^\nu\frac{1}{2T}\partial_\nu T g^{\rho\sigma}\delta_{\mathcal{B}}g_{\rho\sigma} + \delta_{\mathcal{B}}\left(f_1\zeta^\nu\frac{1}{T}\partial_\nu T\right) - \nabla_\mu\left(f_1\zeta^\mu\frac{1}{T}\delta_{\mathcal{B}}T\right)\\
&= f_1\zeta^\nu\frac{1}{2T}\partial_\nu T P^{\rho\sigma}\delta_{\mathcal{B}}g_{\rho\sigma} + f_1\frac{1}{T}\partial_\nu T\delta_{\mathcal{B}}\zeta^\nu + \zeta^\nu\frac{1}{T}\partial_\nu T\delta_{\mathcal{B}}f_1\\
&\quad - f_1\zeta^\nu\frac{1}{2T}\partial_\nu T u^\rho u^\sigma\delta_{\mathcal{B}}g_{\rho\sigma} - f_1\zeta^\nu\frac{1}{T^2}\partial_\nu T\delta_{\mathcal{B}}T + f_1\zeta^\nu\frac{1}{T}\partial_\nu\delta_{\mathcal{B}}T - \nabla_\mu\left(f_1\zeta^\mu\frac{1}{T}\delta_{\mathcal{B}}T\right)\\
&= f_1\zeta^\nu\frac{1}{2T}\partial_\nu T P^{\rho\sigma}\delta_{\mathcal{B}}g_{\rho\sigma} + f_1\frac{1}{T}\partial_\nu T[(u^\nu u^{(\rho}\zeta^{\sigma)} - P^{\nu(\rho}\xi^{\sigma)})\delta_{\mathcal{B}}g_{\rho\sigma} + P^{\nu\rho}\delta_{\mathcal{B}}\xi_\rho]\\
&\quad + \zeta^\nu\frac{1}{T}\partial_\nu T\left(\frac{\partial f_1}{\partial T}\delta_{\mathcal{B}}T + \frac{\partial f_1}{\partial\nu}\delta_{\mathcal{B}}\nu + \frac{\partial f_1}{\partial\hat{\mu}_s}\delta_{\mathcal{B}}\hat{\mu}_s\right)\\
&\quad - f_1\zeta^\nu\frac{1}{2T}\partial_\nu T u^\rho u^\sigma\delta_{\mathcal{B}}g_{\rho\sigma} - \nabla_\mu(f_1\zeta^\mu)\frac{1}{T}\delta_{\mathcal{B}}T\\
&= \left[u^\rho u^\sigma\left(\alpha_{E,1}S_{e,1} - \frac{1}{T}\nabla_\mu(Tf_1\zeta^\mu)\right) + (\zeta^\rho\zeta^\sigma - 2(u^\mu\xi_\mu)u^{(\rho}\zeta^{\sigma)})S_{e,1}\alpha_{R_s,1}\right.\\
&\quad \left. + \tilde{P}^{\rho\sigma}f_1S_{e,1} + 2u^{(\rho}\zeta^{\sigma)}f_1S_5 - f_12\xi^{(\rho}V_{e,1}^{\sigma)}\right]\frac{1}{2}\delta_{\mathcal{B}}g_{\rho\sigma}\\
&\quad + [u^\rho\alpha_{Q,1}S_{e,1} + f_1V_{e,1}^\rho - \zeta^\rho\alpha_{R_s,1}S_{e,1}]\delta_{\mathcal{B}}A_\rho + [f_1V_{e,1}^\rho - \zeta^\rho\alpha_{R_s,1}S_{e,1}]\partial_\rho\delta_{\mathcal{B}}\varphi.
\end{aligned}\tag{C6}$$

Performing a differentiation by parts,

$$\begin{aligned}
\nabla_\mu\left(2f_1 u^{[\mu}\zeta^{\nu]}\frac{1}{T^2}\partial_\nu T + \mathcal{O}(\partial^2)\right) &= \left[u^\rho u^\sigma\left(\alpha_{E,1}S_{e,1} - \frac{1}{T}\nabla_\mu(Tf_1\zeta^\mu)\right) + (\zeta^\rho\zeta^\sigma - 2(u^\mu\xi_\mu)u^{(\rho}\zeta^{\sigma)})S_{e,1}\alpha_{R_s,1}\right.\\
&\quad \left. + \tilde{P}^{\rho\sigma}f_1S_{e,1} + 2u^{(\rho}\zeta^{\sigma)}f_1S_5 - f_12\xi^{(\rho}V_{e,1}^{\sigma)}\right]\frac{1}{2}\delta_{\mathcal{B}}g_{\rho\sigma}\\
&\quad + [u^\rho\alpha_{Q,1}S_{e,1} + f_1V_{e,1}^\rho - \zeta^\rho\alpha_{R_s,1}S_{e,1}]\delta_{\mathcal{B}}A_\rho - \nabla_\rho[f_1V_{e,1}^\rho - \zeta^\rho\alpha_{R_s,1}S_{e,1}]\delta_{\mathcal{B}}\varphi.
\end{aligned}\tag{C7}$$

From here we can read out the contributions to the constitutive relations Eq. (A13). Similarly divergence of the other term in Eq. (A11) coupling to f_2 can also be computed. Now, the first-order parity-odd free energy current $\mathcal{N}^\mu$ in Eq. (A23) has a term $g_2\beta^\mu\tilde{S}_{e,2} + g_2\tilde{V}_2^\mu$. We can compute its divergence as

$$\begin{aligned}
\nabla_\mu(g_2\beta^\mu\tilde{S}_{e,2} + g_2\tilde{V}_2^\mu) &= \epsilon^{\tau\nu\rho\sigma}\delta_{\mathcal{B}}\left(g_2T\frac{1}{2}\xi_\tau u_\nu F_{\rho\sigma}\right) - \nabla_\mu(\epsilon^{\mu\tau\nu\sigma}g_2T\xi_\tau u_\nu\delta_{\mathcal{B}}A_\sigma)\\
&= \frac{T}{2}\epsilon^{\tau\nu\rho\sigma}\xi_\tau u_\nu F_{\rho\sigma}\delta_{\mathcal{B}}g_2 + \frac{1}{2}g_2\epsilon^{\tau\nu\rho\sigma}\xi_\tau u_\nu F_{\rho\sigma}\delta_{\mathcal{B}}T + \epsilon^{\tau\nu\rho\sigma}g_2T\frac{1}{2}\xi_\tau F_{\rho\sigma}\delta_{\mathcal{B}}u_\nu\\
&\quad + \frac{T}{2}g_2\epsilon^{\tau\nu\rho\sigma}u_\nu F_{\rho\sigma}\delta_{\mathcal{B}}\xi_\tau + \epsilon^{\tau\nu\rho\sigma}g_2T\xi_\tau u_\nu\nabla_\rho\delta_{\mathcal{B}}A_\sigma - \nabla_\rho(\epsilon^{\tau\nu\rho\sigma}g_2T\xi_\tau u_\nu\delta_{\mathcal{B}}A_\sigma)\\
&= \frac{T}{2}\epsilon^{\tau\nu\rho\sigma}\xi_\tau u_\nu F_{\rho\sigma}\left(\frac{\partial g_2}{\partial T}\delta_{\mathcal{B}}T + \frac{\partial g_2}{\partial\nu}\delta_{\mathcal{B}}\nu + \frac{\partial g_2}{\partial\hat{\mu}_s}\delta_{\mathcal{B}}\hat{\mu}_s\right) + \epsilon^{\tau\nu\rho\sigma}g_2T\frac{1}{2}\xi_\tau F_{\rho\sigma}2P_\nu^{(\rho}u^{\sigma)}\frac{1}{2}\delta_{\mathcal{B}}g_{\rho\sigma}\\
&\quad + \left(g_2T\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}u_\nu F_{\rho\sigma} - \nabla_\rho(\epsilon^{\rho\mu\tau\nu}g_2T\xi_\tau u_\nu)\right)\delta_{\mathcal{B}}A_\mu + g_2T\frac{1}{2}\epsilon^{\tau\nu\rho\sigma}u_\nu F_{\rho\sigma}\nabla_\tau\delta_{\mathcal{B}}\varphi\\
&= \left[u^\mu u^\nu\tilde{\alpha}_{E,2}\tilde{S}_{e,2} + 2g_2u^{(\mu}\tilde{V}_{e,4}^{\nu)} + (\zeta^\mu\zeta^\nu - 2(u^\rho\xi_\rho)u^{(\mu}\zeta^{\nu)})\tilde{\alpha}_{R_s,i}\tilde{S}_{e,2} - \zeta^\mu\zeta^\nu\frac{g_2}{2\hat{\mu}_s}\tilde{S}_{e,2}\right]\frac{1}{2}\delta_{\mathcal{B}}g_{\mu\nu}\\
&\quad + [u^\mu\tilde{\alpha}_{Q,2}\tilde{S}_{e,2} + g_2V_{e,2}^\mu - \zeta^\mu\tilde{\alpha}_{R_s,2}\tilde{S}_{e,2} - \nabla_\rho(\epsilon^{\rho\mu\tau\nu}g_2T\xi_\tau u_\nu)]\delta_{\mathcal{B}}A_\mu\\
&\quad + [g_2V_{e,2}^\mu - \zeta^\mu\tilde{\alpha}_{R_s,i}\tilde{S}_{e,2}]\nabla_\mu\delta_{\mathcal{B}}\varphi.
\end{aligned}\tag{C8}$$

Performing a differentiation by parts,

$$
\begin{aligned}
\nabla_\mu(g_2 u^\mu \tilde{S}_{e,2} + g_2 \tilde{V}_2 + \mathcal{O}(\partial^2)) = &\left[u^\mu u^\nu \tilde{\alpha}_{E,2}\tilde{S}_{e,2} + 2g_2 u^{(\mu}\tilde{V}^{\nu)}_{e,4} + (\zeta^\mu\zeta^\nu - 2(u^\rho\xi_\rho)u^{(\mu}\zeta^{\nu)})\tilde{\alpha}_{R_s,i}\tilde{S}_{e,2} - \zeta^\mu\zeta^\nu \frac{g_2}{2\hat{\mu}_s}\tilde{S}_{e,2}\right]\frac{1}{2}\delta_{\mathcal{B}}g_{\mu\nu} \\
&+ [u^\mu \tilde{\alpha}_{Q,2}\tilde{S}_{e,2} + g_2 V^\mu_{e,2} - \zeta^\mu \tilde{\alpha}_{R_s,2}\tilde{S}_{e,2} - \nabla_\rho(\epsilon^{\rho\mu\tau\nu}g_2 T\xi_\tau u_\nu)]\delta_{\mathcal{B}}A_\mu \\
&- \nabla_\mu[g_2 V^\mu_{e,2} - \zeta^\mu\tilde{\alpha}_{R_s,i}\tilde{S}_{e,2}]\delta_{\mathcal{B}}\varphi. \qquad \text{(C9)}
\end{aligned}
$$

From here we can read out the contributions to the constitutive relations Eq. (A24). Similarly divergence of the other term in Eq. (A23) coupling to g_1 can also be computed. There is another term in the parity-odd free energy current $C_1 T^2\omega^\mu$; its divergence is given as

$$
\begin{aligned}
\nabla_\mu(C_1T^2\omega^\mu) &= -2C_1 T\epsilon^{\mu\nu\rho\sigma}u_\mu\partial_\nu T\partial_\rho u_\sigma + C_1T^2\epsilon^{\mu\nu\rho\sigma}\partial_\mu u_\nu\partial_\rho u_\sigma \\
&= 2C_1T^3\omega^{(\mu}u^{\nu)}\delta_{\mathcal{B}}g_{\mu\nu}. \qquad \text{(C10)}
\end{aligned}
$$

This can be matched with the constitutive relations Eq. (A24).

Null superfluid free energy current: We now move on to superfluids. The $\delta_{\mathcal{B}}$ variation of hydrodynamic and superfluid fields can be computed to be,

$$
\begin{aligned}
&\delta_{\mathcal{B}}T = TV^{(M}u^{N)}\delta_{\mathcal{B}}g_{MN}, \qquad \delta_{\mathcal{B}}\nu_n = \frac{1}{2T}u^Mu^N\delta_{\mathcal{B}}g_{MN}, \qquad \delta_{\mathcal{B}}\nu = \frac{1}{T}u^M\delta_{\mathcal{B}}A_M, \\
&\delta_{\mathcal{B}}\mu_s = \frac{1}{2}\xi^M\xi^N\delta_{\mathcal{B}}g_{MN} - \xi^M\delta_{\mathcal{B}}A_M - \xi^M\nabla_M\delta_{\mathcal{B}}\varphi, \\
&\delta_{\mathcal{B}}\hat{\mu}_s = \frac{1}{2}(\zeta^M\zeta^N + 2\zeta^{(M}u^{N)} - 2\zeta^{(M}V^{N)}(u^R\xi_R))\delta_{\mathcal{B}}g_{MN} - \zeta^M\delta_{\mathcal{B}}A_M - \zeta^M\nabla_M\delta_{\mathcal{B}}\varphi, \\
&\delta_{\mathcal{B}}u^M = (2u^MV^{(R}u^{S)} + V^Mu^Ru^S)\frac{1}{2}\delta_{\mathcal{B}}g_{RS}, \qquad \delta_{\mathcal{B}}u_M = (2P^{(R}_M u^{S)} - V_Mu^Ru^S)\frac{1}{2}\delta_{\mathcal{B}}g_{RS}, \\
&\delta_{\mathcal{B}}\zeta^M = (-2\xi^{(R}P^{S)M} + 2\zeta^{(R}V^{S)}u^M + 2\zeta^{(R}u^{S)}V^M)\frac{1}{2}\delta_{\mathcal{B}}g_{RS} + P^{MN}\delta_{\mathcal{B}}\xi_N, \\
&\delta_{\mathcal{B}}\zeta_M = (2(u^N\xi_N)P^{(R}_MV^{S)} - 2P^{(R}_Mu^{S)})\frac{1}{2}\delta_{\mathcal{B}}g_{RS} + P^N_M\delta_{\mathcal{B}}\xi_N. \qquad \text{(C11)}
\end{aligned}
$$

The first-order, parity-even, free-energy current $\mathcal{N}^M$ in Eq. (3.24) has a term $2f_1u^{[M}\zeta^{N]}\frac{1}{T^2}\partial_NT$. We compute its divergence,

$$
\begin{aligned}
\nabla_M\left(2f_1u^{[M}\zeta^{N]}\frac{1}{T^2}\partial_NT\right) &= f_1\zeta^N\frac{1}{2T}\partial_NTg^{RS}\delta_{\mathcal{B}}g_{RS} + \delta_{\mathcal{B}}\left(f_1\frac{1}{T}\zeta^N\partial_NT\right) - \nabla_M\left(f_1\zeta^M\frac{1}{T}\delta_{\mathcal{B}}T\right) \\
&= f_1\zeta^N\frac{1}{2T}\partial_NTP^{RS}\delta_{\mathcal{B}}g_{RS} + f_1\frac{1}{T}\partial_NT\delta_{\mathcal{B}}\zeta^N + \frac{1}{T}\zeta^N\partial_NT\delta_{\mathcal{B}}f_1 \\
&\quad - f_1\zeta^N\frac{1}{T}\partial_NTV^Ru^S\delta_{\mathcal{B}}g_{RS} - f_1\frac{1}{T^2}\zeta^N\partial_NT\delta_{\mathcal{B}}T + f_1\frac{1}{T}\zeta^N\partial_N\delta_{\mathcal{B}}T - \nabla_M\left(f_1\zeta^M\frac{1}{T}\delta_{\mathcal{B}}T\right) \\
&= f_1\zeta^N\frac{1}{2T}\partial_NTP^{RS}\delta_{\mathcal{B}}g_{RS} + f_1\frac{1}{T}\partial_MT\left[(-2\xi^{(R}P^{S)M} + 2\zeta^{(R}V^{S)}u^M)\frac{1}{2}\delta_{\mathcal{B}}g_{RS} + P^{MN}\delta_{\mathcal{B}}\xi_N\right] \\
&\quad + \frac{1}{T}\zeta^N\partial_NT\left(\frac{\partial f_1}{\partial T}\delta_{\mathcal{B}}T + \frac{\partial f_1}{\partial\nu}\delta_{\mathcal{B}}\nu + \frac{\partial f_1}{\partial\nu_n}\delta_{\mathcal{B}}\nu_n + \frac{\partial f_1}{\partial\hat{\mu}_s}\delta_{\mathcal{B}}\hat{\mu}_s\right) \\
&\quad - f_1\zeta^N\frac{1}{T}\partial_NTV^Ru^S\delta_{\mathcal{B}}g_{RS} - \nabla_M(f_1\zeta^M)\frac{1}{T}\delta_{\mathcal{B}}T \\
&= \Bigg[2V^{(R}u^{S)}\left(\alpha_{E,1}S_{e,1} - \frac{1}{T}\nabla_M(Tf_1\zeta^M)\right) + u^Ru^S\alpha_{R_n,1}S_{e,1} + \tilde{P}^{RS}f_1S_{e,1} - 2f_1\xi^{(R}V^{S)}_{e,1} \\
&\quad + (\zeta^R\zeta^S + 2\zeta^{(R}u^{S)} - 2\zeta^{(R}V^{S)}(u^M\xi_M))\alpha_{R_s,1}S_{e,1} + 2\zeta^{(R}V^{S)}f_1S_6\Bigg]\frac{1}{2}\delta_{\mathcal{B}}g_{RS} \\
&\quad + [u^M\alpha_{Q,1}S_{e,1} - \zeta^M\alpha_{R_s,1}S_{e,1} + f_1V^M_{e,1}]\delta_{\mathcal{B}}A_M + [f_1V^M_{e,1} - \zeta^M\alpha_{R_s,1}S_{e,1}]\nabla_M\delta_{\mathcal{B}}\varphi. \qquad \text{(C12)}
\end{aligned}
$$

Performing a differentiation by parts,

$$\nabla_M\left(2f_1u^{[M}\zeta^{N]}\frac{1}{T^2}\partial_N T+\mathcal{O}(\partial^2)\right)=\Big[2V^{(R}u^{S)}\left(\alpha_{E,1}S_{e,1}-\frac{1}{T}\nabla_M(Tf_1\zeta^M)\right)+u^Ru^S\alpha_{R_n,1}S_{e,1}+\tilde{P}^{RS}f_1S_{e,1}-2f_1\xi^{(R}V^{S)}_{e,1}$$
$$+(\zeta^R\zeta^S+2\zeta^{(R}u^{S)}-2\zeta^{(R}V^{S)}(u^M\xi_M))\alpha_{R_s,1}S_{e,1}+2\zeta^{(R}V^{S)}f_1S_6\Big]\frac{1}{2}\delta_{\mathcal{B}}g_{RS}$$
$$+[u^M\alpha_{Q,1}S_{e,1}-\zeta^M\alpha_{R_s,1}S_{e,1}+f_1V^M_{e,1}]\delta_{\mathcal{B}}A_M+\nabla_M[\zeta^M\alpha_{R_s,1}S_{e,1}-f_1V^M_{e,1}]\delta_{\mathcal{B}}\varphi. \tag{C13}$$

From here we can read out the contributions to the constitutive relations Eq. (3.26). Similarly divergence of the other terms in Eq. (3.24) coupling to f_2, f_3 can also be computed. Now, the first-order, parity-odd, free-energy current $\mathcal{N}^M$ in Eq. (3.36) has a term $g_2\beta^M\tilde{S}_{e,2}+g_2\tilde{V}^M_3$. We can compute its divergence as

$$\begin{aligned}
\nabla_M(g_2\beta^M\tilde{S}_{e,2}+g_2\tilde{V}^M_3)&=\frac{1}{2}\epsilon^{NRSTK}\delta_{\mathcal{B}}(g_2T\xi_NV_Ru_SF_{TK})-\nabla_T(\epsilon^{NRSTK}g_2T\xi_NV_Ru_S\delta_{\mathcal{B}}A_K)\\
&=\frac{1}{2}\epsilon^{NRSTK}T\xi_NV_Ru_SF_{TK}\delta_{\mathcal{B}}g_2+\frac{1}{2}\epsilon^{NRSTK}g_2T\xi_NV_RF_{TK}\delta_{\mathcal{B}}u_S+\frac{1}{2}\epsilon^{NRSTK}g_2\xi_NV_Ru_SF_{TK}\delta_{\mathcal{B}}T\\
&\quad+\frac{1}{2}\epsilon^{NRSTK}g_2T\xi_Nu_SF_{TK}\delta_{\mathcal{B}}V_R+\frac{1}{2}\epsilon^{NRSTK}g_2TV_Ru_SF_{TK}\delta_{\mathcal{B}}\xi_N\\
&\quad+\epsilon^{NRSTK}g_2T\xi_NV_Ru_S\nabla_T\delta_{\mathcal{B}}A_K-\nabla_T(\epsilon^{NRSTK}g_2T\xi_NV_Ru_S\delta_{\mathcal{B}}A_K)\\
&=\frac{1}{2}\epsilon^{NRSTK}T\xi_NV_Ru_SF_{TK}\left(\frac{\partial g_2}{\partial T}\delta_{\mathcal{B}}T+\frac{\partial g_2}{\partial\nu}\delta_{\mathcal{B}}\nu+\frac{\partial g_2}{\partial\nu_n}\delta_{\mathcal{B}}\nu_n+\frac{\delta g_2}{\delta\hat{\mu}_s}\delta_{\mathcal{B}}\hat{\mu}_s\right)\\
&\quad-u^AP^B_Mg_2T\frac{1}{2}\epsilon^{MNRTK}V_Nu_RF_{TK}\delta_{\mathcal{B}}g_{AB}+\frac{1}{2}\epsilon^{NRSTK}g_2T\xi_Nu_SF_{TK}P^B_RV^A\delta_{\mathcal{B}}g_{AB}\\
&\quad-\nabla_T(\epsilon^{TMNRS}g_2T\xi_NV_Ru_S)\delta_{\mathcal{B}}A_M+\frac{1}{2}\epsilon^{NRSTK}g_2TV_Ru_SF_{TK}\delta_{\mathcal{B}}\xi_N\\
&=\Big[2\tilde{\alpha}_{E,2}V^{(M}u^{N)}\tilde{S}_{e,2}+\tilde{\alpha}_{R_n,2}u^Mu^N\tilde{S}_{e,2}-2g_2u^{(M}\tilde{V}^{N)}_{e,2}-2g_2V^{(M}\tilde{V}^{N)}_{e,4}-\zeta^M\zeta^N\frac{g_2}{2\hat{\mu}_s}\tilde{S}_{e,2}\\
&\quad+(\zeta^M\zeta^N+2\zeta^{(M}u^{N)}-2\zeta^{(M}V^{N)}(u^R\xi_R))\tilde{\alpha}_{R_s,2}\tilde{S}_{e,2}\Big]\frac{1}{2}\delta_{\mathcal{B}}g_{MN}\\
&\quad+[u^M\tilde{\alpha}_{Q,2}\tilde{S}_{e,2}+g_2\tilde{V}_{e,2}-\zeta^M\tilde{\alpha}_{R_s,2}\tilde{S}_{e,2}-P^M_K\nabla_T(\epsilon^{TKNRS}g_2T\xi_NV_Ru_S)]\delta_{\mathcal{B}}A_M\\
&\quad+[g_2\tilde{V}_{e,2}-\zeta^M\tilde{\alpha}_{R_s,2}\tilde{S}_{e,2}]\nabla_M\delta_{\mathcal{B}}\varphi.
\end{aligned}\tag{C14}$$

Performing a differentiation by parts,

$$\nabla_M(g_2\beta^M\tilde{S}_{e,2}+g_2\tilde{V}^M_3+\mathcal{O}(\partial^2))=\Big[2V^{(M}u^{N)}\tilde{\alpha}_{E,2}\tilde{S}_{e,2}+u^Mu^N\tilde{\alpha}_{R_n,2}\tilde{S}_{e,2}-2g_2u^{(M}\tilde{V}^{N)}_{e,2}-2g_2V^{(M}\tilde{V}^{N)}_{e,4}-\zeta^M\zeta^N\frac{g_2}{2\hat{\mu}_s}\tilde{S}_{e,2}$$
$$+(\zeta^M\zeta^N+2\zeta^{(M}u^{N)}-2\zeta^{(M}V^{N)}(u^R\xi_R))\tilde{\alpha}_{R_s,2}\tilde{S}_{e,2}\Big]\frac{1}{2}\delta_{\mathcal{B}}g_{MN}$$
$$+[u^M\tilde{\alpha}_{Q,2}\tilde{S}_{e,2}+g_2\tilde{V}_{e,2}-\zeta^M\tilde{\alpha}_{R_s,2}\tilde{S}_{e,2}-P^M_K\nabla_T(\epsilon^{TKNRS}g_2T\xi_NV_Ru_S)]\delta_{\mathcal{B}}A_M$$
$$+\nabla_M[\zeta^M\tilde{\alpha}_{R_s,2}\tilde{S}_{e,2}-g_2\tilde{V}_{e,2}]\delta_{\mathcal{B}}\varphi. \tag{C15}$$

From here, we can read out the contributions to the constitutive relations, Eq. (3.37). Similarly, divergence of the other term in Eq. (3.36) coupling to g_1 can also be computed. Divergence of the term coupling to g_3 is particularly simple,

$$\nabla_M(g_3\tilde{V}_1^M) = \nabla_M\left(\frac{g_3}{T}\epsilon^{MNRST}V_N u_R \zeta_S \partial_T T\right) = -\nabla_M(g_3 T\epsilon^{MNRST}V_N u_R\zeta_S)\partial_T\left(\frac{1}{T}\right)$$
$$= V^{(M}P_P^{N)}\nabla_K(g_3 T\epsilon^{PKRST}V_R u_S \zeta_T)\delta_{\mathcal{B}}g_{MN}. \tag{C16}$$

Finally the last term in parity-odd free energy current $C_1 T\omega^M$ has divergence,

$$\nabla_M(C_1 T\omega^M) = C_1 T^2 \omega^{(M}V^{N)}\delta_{\mathcal{B}}g_{MN}. \tag{C17}$$

This can be matched with the constitutive relations in Eq. (3.37).

[1] P. Kapitza, Viscosity of liquid helium below the λ-point, Nature (London) **141**, 74 (1938).
[2] J. F. Allen and A. D. Misener, Flow phenomena in liquid helium II, Nature (London) **142**, 643 (1938).
[3] F. London, The λ-phenomenon of liquid helium and the Bose-Einstein degeneracy, Nature (London) **141**, 643 (1938).
[4] L. Landau, Theory of the superfluidity of helium II, Phys. Rev. **60**, 356 (1941).
[5] L. Tisza, The theory of liquid helium, Phys. Rev. **72**, 838 (1947).
[6] I. M. Khalatnikov and V. V. Lebedev, Relativistic hydrodynamics of a superfluid liquid, Phys. Lett. **91A**, 70 (1982).
[7] W. Israel, Covariant superfluid mechanics, Phys. Lett. **86A**, 79 (1981).
[8] W. Israel, Equivalence of two theories of relativistic superfluid mechanics, Phys. Lett. **92A**, 77 (1982).
[9] B. Carter and I. M. Khalatnikov, Equivalence of convective and potential variational derivations of covariant superfluid dynamics, Phys. Rev. D **45**, 4536 (1992).
[10] B. Carter and I. M Khalatnikov, Momentum, vorticity, and helicity in covariant superfluid dynamics, Ann. Phys. (N.Y.) **219**, 243 (1992).
[11] D. T. Son, Hydrodynamics of relativistic systems with broken continuous symmetries, Int. J. Mod. Phys. A **16**, 1284 (2001).
[12] J. Bhattacharya and S. Bhattacharyya, and S. Minwalla, Dissipative superfluid dynamics from gravity, J. High Energy Phys. 04 (2011) 125.
[13] J. Bhattacharya, S. Bhattacharyya, S. Minwalla, and A. Yarom, A theory of first order dissipative superfluid dynamics, J. High Energy Phys. 05 (2014) 147.
[14] S. Bhattacharyya, S. Jain, S. Minwalla, and T. Sharma, Constraints on superfluid hydrodynamics from equilibrium partition functions, J. High Energy Phys. 01 (2013) 040.
[15] V. P. Kirilin, A. V. Sadofyev, and V. I. Zakharov, Chiral vortical effect in superfluid, Phys. Rev. D **86**, 025021 (2012).
[16] S. Giorgini, L. P. Pitaevskii, and S. Stringari, Theory of ultracold atomic Fermi gases, Rev. Mod. Phys. **80**, 1215 (2008).
[17] N. Banerjee, S. Dutta, and A. J. Akash, Null fluids—A new viewpoint of Galilean fluids, Phys. Rev. D **93**, 105020 (2016).
[18] N. Banerjee, S. Dutta, and A. Jain, Equilibrium partition function for nonrelativistic fluids, Phys. Rev. D **92**, 081701 (2015).
[19] N. Banerjee, S. Dutta, A. Jain, and D. Roychowdhury, Entropy current for non-relativistic fluid, J. High Energy Phys. 08 (2014) 037.
[20] M. Rangamani, S. F. Ross, D. T. Son, and E. G. Thompson, Conformal non-relativistic hydrodynamics from gravity, J. High Energy Phys. 01 (2009) 075.
[21] M. Hassaine and P. A. Horvathy, Field dependent symmetries of a nonrelativistic fluid model, Ann. Phys. (N.Y.) **282**, 218 (2000).
[22] A. Jain, Galilean anomalies and their effect on hydrodynamics, Phys. Rev. D **93**, 065007 (2016).
[23] L. D. Landau and E. M. Lifshitz, Fluid Mechanics. *Teoreticheskaia fizika* (Pergamon Press, New York, 1959).
[24] R. Loganayagam, Anomaly induced transport in arbitrary dimensions, arXiv:1106.0277.
[25] F. M. Haehl, R. Loganayagam, and M. Rangamani, The Eightfold Way to Dissipation, Phys. Rev. Lett. **114**, 201601 (2015).
[26] F. M. Haehl, R. Loganayagam, and M. Rangamani, Adiabatic hydrodynamics: The eightfold way to dissipation, J. High Energy Phys. 05 (2015) 060.
[27] A. Jain, A theory for non-Abelian superfluid dynamics, Phys. Rev. D **95**, 121701 (2017).
[28] D. T. Son and P. Surowka, Hydrodynamics with Triangle Anomalies, Phys. Rev. Lett. **103**, 191601 (2009).
[29] N. Banerjee, J. Bhattacharya, S. Bhattacharyya, S. Jain, S. Minwalla, and S. Tarun, Constraints on fluid dynamics from equilibrium partition functions, J. High Energy Phys. 09 (2012) 046.
[30] K. Jensen, M. Kaminski, P. Kovtun, R. Meyer, A. Ritz, and A. Yarom, Towards Hydrodynamics without an Entropy Current, Phys. Rev. Lett. **109**, 101601 (2012).
[31] S. Bhattacharyya, Entropy current from partition function: One example, J. High Energy Phys. 07 (2014) 139.
[32] S. Bhattacharyya, Entropy current and equilibrium partition function in fluid dynamics, J. High Energy Phys. 08 (2014) 165.
[33] S. Bhattacharyya, Constraints on the second order transport coefficients of an uncharged fluid, J. High Energy Phys. 07 (2012) 104.

[34] N. Banerjee, S. Dutta, and A. Jain, Higher derivative corrections to charged fluids in 2n dimensions, J. High Energy Phys. 05 (2015) 010.
[35] M. H. Christensen, J. Hartong, N. A. Obers, and B. Rollier, Torsional Newton-Cartan geometry and Lifshitz holography, Phys. Rev. D **89**, 061901 (2014).
[36] K. Jensen, On the coupling of Galilean-invariant field theories to curved spacetime, arXiv:1408.6855.
[37] M. Kaminski and S. Moroz, Nonrelativistic parity-violating hydrodynamics in two spatial dimensions, Phys. Rev. B **89**, 115418 (2014).
[38] K. Jensen and A. Karch, Revisiting non-relativistic limits, J. High Energy Phys. 04 (2015) 155.
[39] S. Grozdanov and J. Polonyi, Viscosity and dissipative hydrodynamics from effective field theory, Phys. Rev. D **91**, 105031 (2015).
[40] P. Kovtun, G. D. Moore, and P. Romatschke, Towards an effective action for relativistic dissipative hydrodynamics, J. High Energy Phys. 07 (2014) 123.
[41] M. Harder, P. Kovtun, and A. Ritz, On thermal fluctuations and the generating functional in relativistic hydrodynamics, J. High Energy Phys. 07 (2015) 025.
[42] F. M. Haehl, R. Loganayagam, and M. Rangamani, Topological sigma models & dissipative hydrodynamics, J. High Energy Phys. 04 (2016) 039.
[43] M. Crossley, P. Glorioso, H. Liu, and Y. Wang, Off-shell hydrodynamics from holography, J. High Energy Phys. 02 (2016) 124.
[44] J. de Boer, M. P. Heller, and N. Pinzani-Fokeeva, Effective actions for relativistic fluids from holography, J. High Energy Phys. 08 (2015) 086.
[45] J. Armas, J. Bhattacharya, and N. Kundu, Surface transport in plasma-balls, J. High Energy Phys. 06 (2016) 015.
[46] J. Armas, J. Bhattacharya, A. Jain, and N. Kundu, On the surface of superfluids, J. High Energy Phys. 06 (2017) 090.
[47] N. Banerjee, S. Bhatkar, and J. Akash, Second order galilean fluid dynamics (to be published).